DRAFT VERSION OCTOBER 21, 2009
Preprint typeset using LaTeX style emulateapj v. 4/12/04

THE HALO MASS FUNCTION FROM EXCURSION SET THEORY. III. NON-GAUSSIAN FLUCTUATIONS

MICHELE MAGGIORE[1] AND ANTONIO RIOTTO[2,3]

Draft version October 21, 2009

ABSTRACT

We compute the effect of primordial non-Gaussianity on the halo mass function, using excursion set theory. In the presence of non-Gaussianity the stochastic evolution of the smoothed density field, as a function of the smoothing scale, is non-markovian and beside "local" terms that generalize Press-Schechter (PS) theory, there are also "memory" terms, whose effect on the mass function can be computed using the formalism developed in the first paper of this series. We find that, when computing the effect of the three-point correlator on the mass function, a PS-like approach which consists in neglecting the cloud-in-cloud problem and in multiplying the final result by a fudge factor $\simeq 2$, is in principle not justified. When computed correctly in the framework of excursion set theory, in fact, the "local" contribution vanishes (for all odd-point correlators the contribution of the image gaussian cancels the Press-Schechter contribution rather than adding up), and the result comes entirely from non-trivial memory terms which are absent in PS theory. However it turns out that, in the limit of large halo masses, where the effect of non-Gaussianity is more relevant, these memory terms give a contribution which is the the same as that computed naively with PS theory, plus subleading terms depending on derivatives of the three-point correlator. We finally combine these results with the diffusive barrier model developed in the second paper of this series, and we find that the resulting mass function reproduces recent N-body simulations with non-Gaussian initial conditions, without the introduction of any ad hoc parameter.

Subject headings: cosmology:theory — dark matter:halos — large scale structure of the universe

1. INTRODUCTION

In the first two papers of this series (Maggiore & Riotto (2009a) and Maggiore & Riotto (2009b), papers I and II in the following) we have studied the mass function of dark matter halos using the excursion set formalism. The halo mass function can be written as

$$\frac{dn(M)}{dM} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}(M)}{d\ln M}\,, \tag{1}$$

where $n(M)$ is the number density of dark matter halos of mass M, $\sigma(M)$ is the variance of the linear density field smoothed on a scale R corresponding to a mass M, and $\bar{\rho}$ is the average density of the universe. Analytical derivations of the halo mass function are typically based on Press-Schechter (PS) theory (Press & Schechter 1974) and its extension (Peacock & Heavens 1990; Bond et al. 1991) known as excursion set theory (see Zentner (2007) for a recent review). In excursion set theory the density perturbation evolves stochastically with the smoothing scale, and the problem of computing the probability of halo formation is mapped into the so-called first-passage time problem in the presence of a barrier. With this method, for gaussian fluctuations one obtains

$$f_{\rm PS}(\sigma) = \left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)}\,, \tag{2}$$

where $\delta_c \simeq 1.686$ is the critical value in the spherical collapse model. This result can be extended to arbitrary redshift z by reabsorbing the evolution of the variance into δ_c, so that δ_c in the above result is replaced by $\delta_c(z) = \delta_c(0)/D(z)$, where $D(z)$ is the linear growth factor. Equation (2) is only valid when the density contrast is smoothed with a sharp filter in momentum space. In this case the evolution of the density contrast $\delta(R)$ with the smoothing scale is markovian, and the probability that the density contrast reaches a given value δ at a given smoothing scale satisfies a Fokker-Planck equation with an "absorbing barrier" boundary condition. From the solution of this equation one obtains eq. (2), including a well-known factor of two that Press and Schecther were forced to add by hand.

However, as is well-known, a sharp filter in momentum space is not appropriate for comparison with experimental data from upcoming galaxy surveys, nor with N-body simulations, because it is not possible to associate unambiguously a mass M to the smoothing scale R used in this filter. Rather, one should use a tophat filter in coordinate space, in which case the mass associated to a smoothing scale R is trivially $(4/3)\pi R^3\rho$. If one wants to compute the halo mass function with a tophat filter in coordinates space one is confronted with a much more difficult problem, where the evolution of δ with the smoothing scale is no longer markovian (Bond et al. 1991). Nevertheless, in paper I we succeeded in developing a formalism that allows us to compute perturbatively these non-markovian effects and, for gaussian fluctuations, we found that, to first order, eq. (2) is modified to

$$f(\sigma) = (1-\kappa)\left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)} + \frac{\kappa}{\sqrt{2\pi}}\frac{\delta_c}{\sigma}\,\Gamma\left(0,\frac{\delta_c^2}{2\sigma^2}\right)\,, \tag{3}$$

where

$$\kappa(R) \equiv \lim_{R'\to\infty}\frac{\langle\delta(R')\delta(R)\rangle}{\langle\delta^2(R')\rangle} - 1 \simeq 0.4592 - 0.0031\,R\,, \tag{4}$$

R is measured in Mpc$/h$, $\Gamma(0,z)$ is the incomplete Gamma function, and the numerical value of $\kappa(R)$ is computed using a tophat filter function in coordinate space and a ΛCDM model with $\sigma_8 = 0.8$, $h = 0.7$, $\Omega_M = 1-\Omega_\Lambda = 0.28$, $\Omega_B = 0.046$ and $n_s = 0.96$, consistent with the WMAP 5-years data release.

[1] Département de Physique Théorique, Université de Genève, 24 quai Ansermet, CH-1211 Genève, Switzerland
[2] CERN, PH-TH Division, CH-1211, Genève 23, Switzerland
[3] INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padua, Italy

This analytical result reproduces well the result of a Monte Carlo realization of the first-crossing distribution of excursion set theory, obtained by integrating numerically a Langevin equation with a colored noise, performed in Bond et al. (1991) and in Robertson et al. (2008). This is a useful test of our technique. Still, neither eq. (2) nor eq. (3) perform well when compared to cosmological N-body simulation, which means that some crucial physical ingredient is still missing in the model. This is not surprising, since the spherical (or ellipsoidal) collapse model is a very simplified description of the process of halo formation which, as shown by N-body simulations, is much more complicated, and proceeds through a mixture of smooth accretion and violent encounters leading to merging as well as to fragmentation (see Springel et al. (2005) and the related movies at http://www.mpa-garching.mpg.de/galform/millennium/). Furthermore, the very operative definition of what is a dark matter halo is a subtle issue. Real halos are not spherical. They are at best triaxial, and often much more irregular, expecially if they experienced recent mergers. Searching for them with a spherical overdensity algorithm therefore introduces further statistical uncertainties. Similar considerations hold for Friends-of-Friends halo finders.

In paper II we have discussed in detail these uncertainties and we have proposed that at least some of the complications intrinsic to a realistic process of halo formation (as well as the statistical uncertainties related to the details of the halo finder) can be accounted for, within the excursion set framework, by treating the critical value for collapse as a stochastic variable. In this approach all our ignorance on the details of halo formation is buried into the variance of the fluctuations of the collapse barrier. The computation of the halo mass function is then mapped into a first-passage time process in the presence of a diffusing barrier, i.e. a barrier whose height evolves according to a diffusion equation. For gaussian fluctuations we found that eq. (3) must be replaced by

$$f(\sigma) = (1-\tilde{\kappa}) \left(\frac{2}{\pi}\right)^{1/2} \frac{a^{1/2}\delta_c}{\sigma}\, e^{-a\delta_c^2/(2\sigma^2)} + \frac{\tilde{\kappa}}{\sqrt{2\pi}} \frac{a^{1/2}\delta_c}{\sigma} \Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right), \tag{5}$$

where

$$a = \frac{1}{1+D_B}, \qquad \tilde{\kappa} = \frac{\kappa}{1+D_B}, \tag{6}$$

and D_B is an effective diffusion coefficient for the barrier. A first-principle computation of D_B appears difficult, but from recent studies of the properties of the collapse barrier in N-body simulations (Robertson et al. 2008) we deduced a value $D_B \simeq (0.3\delta_c)^2$. Using this value for D_B in eq. (6) gives $a \simeq 0.80$, so

$$\sqrt{a} \simeq 0.89. \tag{7}$$

We see that the net effect of the diffusing barrier is that, in the mass function, δ_c is replaced by $a^{1/2}\delta_c$, which is the replacement that was made by hand, simply to fit the data, in Sheth & Tormen (1999); Sheth et al. (2001).

The above result was obtained by considering a barrier that fluctuates over the constant value δ_c of the spherical collapse model. More generally, one should consider fluctuations over the barrier $B(\sigma)$ given by the ellipsoidal collapse model. Since the latter reduces to the former in the small σ limit (i.e. for large halo masses), eq. (5) is better seen as the large mass limit of a more accurate mass function obtained from a barrier that flucutates around the average value $B(\sigma)$ given by the ellipsoidal collapse model. When $\kappa = 0$ eq. (5) reduces to the large mass limit of the Sheth-Tormen mass function. So, eq. (5) generalizes the Sheth-Tormen mass function by taking into account the effect of the tophat filter in coordinate space, while eq. (6) provides a physical motivation for the introduction of the parameter a.

Equation (5) is in excellent agreement with the N-body simulations for gaussian primordial fluctuations, see Figs. 6 and 7 of paper II. We stress that our value $a \simeq 0.80$ is not determined by fitting the mass function to the data. We do have an input from the N-body simulation here, which is however quite indirect, and is the measured variance of the threshold for collapse, which for small σ is determined in Robertson et al. (2008) to be $\Sigma_B \simeq 0.3\sigma$. Our diffusing barrier model of paper II translates this information into an effective diffusion coefficient for the barrier, $D_B = (0.3\delta_c)^2$, and predicts $a = 1/(1+D_B)$. We refer the reader to paper II for details and discussions of the physical motivations for the introduction of a stochastic barrier.

The above results refer to initial density fluctuations which have a gaussian distribution. In this paper we attack the problem of the effect on the halo mass function of non-Gaussianities in the primordial density field. Over the last decade a great deal of evidence has been accumulated from the Cosmic Microwave Background (CMB) anisotropy and Large Scale Structure (LSS) spectra that the observed structures originated from seed fluctuations generated during a primordial stage of inflation. While standard one-single field models of slow-roll inflation predict that these fluctuations are very close to gaussian (see Acquaviva et al. (2003); Maldacena (2003)), non-standard scenarios allow for a larger level of non-Gaussianity (see Bartolo et al. (2004) and refs. therein). Deviations from non-Gaussianity are usually parametrized by a dimensionless quantity $f_{\rm NL}$ (Bartolo et al. (2004)) whose value sets the magnitude of the three-point correlation function. If the process generating the primordial non-Gaussianity is local in space, the parameter $f_{\rm NL}$ in Fourier space is independent from the momenta entering the three-point correlation function; if instead the process is non-local in space, like in models of inflation with non-canonical kinetic terms, $f_{\rm NL}$ acquires a dependence on the momenta. It is clear that detecting a significant amount of non-Gaussianity and its shape either from the CMB or from the LSS offers the possibility of opening a window into the dynamics of the universe during the very first stages of its evolution. Current limits on the strength of non-Gaussianity set the $f_{\rm NL}$ parameter to be smaller than $\mathcal{O}(100)$ (Komatsu et al. (2008)).

Non-Gaussianities are particularly relevant in the high-mass end of the power spectrum of perturbations, i.e. on the scale of galaxy clusters, since the effect of non-Gaussian fluctuations becomes especially visible on the tail of the probability distribution. As a result, both the abundance and the clustering properties of very massive halos are sensitive probes of primordial non-Gaussianities (Matarrese et al. 1986; Grinstein & Wise 1986; Lucchin et al. 1988; Moscardini et al. 1991; Koyama et al. 1999; Matarrese et al. 2000; Robinson & Baker 2000; Robinson et al. 2000), and could be detected or significantly constrained by the various planned large-scale galaxy surveys, both ground based (such as DES, PanSTARRS and LSST) and on satellite (such as EUCLID and ADEPT) see, e.g. Dalal et al. (2008) and Carbone et al. (2008). Further-

more, the primordial non-Gaussianity alters the clustering of dark matter halos inducing a scale-dependent bias on large scales (Dalal et al. 2008; Matarrese & Verde 2008; Slosar et al. 2008; Afshordi & Tolley 2008) while even for small primordial non-Gaussianity the evolution of perturbations on super-Hubble scales yields extra contributions on smaller scales (Bartolo et al. (2005)).

At present, there exist already various N-body simulations where non-Gaussianity has been included in the initial conditions (Kang et al. 2007; Grossi et al. 2007; Dalal et al. 2008; Desjacques et al. 2008; Pillepich et al. 2008; Grossi et al. 2009) and which are useful to test the accuracy of the different theoretical predictions for the dark matter halo mass function with non-Gaussianity.

Various attempts at computing analytically the effect of primordial non-Gaussianities on the mass function exist in the literature, based on non-Gaussian extensions of PS theory (Chiu et al. 1997; Robinson & Baker 2000; Matarrese et al. 2000; LoVerde et al. 2008). However, for gaussian flucutations, in the large mass regime PS theory is off by one order of magnitude. It is clear that, by computing non-Gaussian corrections over a theory that, already at the gaussian level, in the relevant regime is off by an order of magnitude, one cannot hope to get the correct mass function for the non-Gaussian case. What is typically done in the recent literature is to take the ratio $R_{\rm NG}(M)$ of the non-Gaussian halo mass function to the gaussian halo mass function, both computed within the framework of PS theory, hoping that even if neither the former nor the latter are correct, still their ratio might catch the main modifications due to non-Gaussianities. The full non-Gaussian halo mass function is then obtained by taking a fit to the data in the gaussian case, such as the Sheth and Thormen mass function (Sheth & Tormen 1999; Sheth et al. 2001), and multiplying it by $R_{\rm NG}(M)$. With this philosophy, the result of Matarrese et al. (2000) reads[4]

$$R_{\rm NG}(\sigma) = \exp\left\{\frac{\delta_c^3\mathcal{S}_3(\sigma)}{6\sigma^2}\right\} \tag{8}$$
$$\times\left|\frac{1}{6}\frac{\delta_c}{\sqrt{1-\delta_c\mathcal{S}_3(\sigma)/3}}\frac{d\mathcal{S}_3}{d\ln\sigma}+\sqrt{1-\delta_c\mathcal{S}_3(\sigma)/3}\right|,$$

where

$$\mathcal{S}_3(\sigma) = \frac{\langle\delta^3(S)\rangle}{\langle\delta^2(S)\rangle^2} \tag{9}$$

is the (normalized) skewness of the density field and, as usual, $S=\sigma^2$ is the variance. Since $\sigma=\sigma(M)$, we can equivalently consider $R_{\rm NG}$ as a function of M.[5]

With a similar philosophy, but a different expansion technique, namely the Edgeworth expansion, LoVerde et al. (2008) propose

$$R_{\rm NG}(\sigma) = 1+\frac{1}{6}\frac{\sigma^2}{\delta_c}\left[\mathcal{S}_3(\sigma)\left(\frac{\delta_c^4}{\sigma^4}-\frac{2\delta_c^2}{\sigma^2}-1\right)+\frac{d\mathcal{S}_3}{d\ln\sigma}\left(\frac{\delta_c^2}{\sigma^2}-1\right)\right]. \tag{10}$$

In the limit $\sigma/\delta_c \ll 1$, eq. (10) becomes

$$R_{\rm NG}(\sigma) = 1+\frac{\delta_c^3\mathcal{S}_3(\sigma)}{6\sigma^2}. \tag{11}$$

The same result is obtained from eq. (8) expanding first to linear order in $\mathcal{S}_3(\sigma)$, and then retaining the leading term of the expansion for small σ/δ_c. The two formulas differ instead at the level of the terms subleading in the expansion for small σ/δ_c. In Grossi et al. (2009), in order to fit the data of N-body simulations, it was suggested to modify both eq. (8) and eq. (10), by making the replacement

$$\delta_c \to \delta_{\rm eff} = \sqrt{a}\,\delta_c\,, \tag{12}$$

with a value $\sqrt{a}\simeq 0.86$ obtained from the fit to the data, very close to our prediction given in eq. (7).[6] In the gaussian case we have shown in paper II that this replacement, which in the previous literature was made ad hoc just to fit the data, actually follows from the diffusive barrier model, see eq. (5), and that the precise value of a depends, among other things, on the details of the halo finder in the simulation, so (slightly) different values of a are obtained from N-body simulations with different halo finders. Below we will see how the results of paper II generalize to the non-Gaussian case.

In Grossi et al. (2009) it is shown that, after performing the replacement (12), both eq. (8) and eq. (10) are in good agreement with the result of N-body simulations with non-Gaussian initial conditions, which a posteriori can be seen as a justification of the procedure used in their derivation. However, it is clear that taking the ratio of two results that, in the interesting mass range, are known to be both off by one order of magnitude, in order to get a fine effect such as the non-Gaussian corrections, can only be considered as a heuristic procedure. First of all, PS theory by itself produces a wrong exponential factor, since it would give $a=1$. Here one might argue that the gaussian and non-Gaussian mass functions have the same exponential behavior, so this effect cancels when considering the ratio $R_{\rm NG}$, and is anyhow accounted for by the heuristic prescription (12). Still, a further source of concern is that the derivation of the PS mass function in Bond et al. (1991) requires that the density field δ evolves with the smoothing scale R (or more precisely with $S(R)$) in a markovian way. Only under this assumption one can derive eq. (2) together with the correct factor of two that Press and Schechter were forced to introduce by hand. As we have discussed at length in paper I, this markovian assumption is broken by the use of a filter function different from a sharp filter in momentum space and, of course, it is further violated by the inclusion of non-Gaussian corrections. When studying non-Gaussianities, it is therefore particularly important to perform the computation including the effect of the filter, otherwise one would attribute to primordial non-Gaussianities effects on the mass functions which are due, more trivially, to the filter.

The formalism that we have developed in papers I and II, however, allows us to attack the problem. First of all, in paper I we have set up a "microscopic" approach which is in principle exact.[7] With this formalism, we computed the non-markovian corrections due to the filter function, which are

[4] We thank S. Matarrese for pointing out to us a typo in Matarrese et al. (2000).

[5] We do not write explicitly the dependence of $R_{\rm NG}(\sigma)$ on redshift z, which enters through the variance σ^2 and, as usual, can be reabsorbed into the height δ_c of the critical value for collapse. The normalized skewness must instead be taken at $z=0$, see Grossi et al. (2009).

[6] As discussed in paper II, the value of the diffusion constant of the barrier D_B, and hence our prediction for a, depends on the halo finder. The value $D_B\simeq 0.25$, which leads to $\sqrt{a}\simeq 0.89$, has been deduced from the simulation of Robertson et al. (2008), that uses a spherical overdensity (SO) halo finder with $\Delta=200$, while Grossi et al. (2009) use a friends-of-friends (FOF) halo finder with link-length 0.2. In the gaussian case, the mass functions obtained from these two finders are very close to each other. However, in order to perform an accurate numerical comparison of our prediction with N-body simulations with non-Gaussian initial conditions, in would be necessary to determine both D_B and the mass function with the same halo finder.

[7] By exact we mean that, given the problem of halo formation as it is formulated within excursion set theory, the path integral technique developed in paper I is an exact way of attacking the mathematical problem of the first-

given by the terms proportional to κ in eq. (3). This is important because it allows us to subtract, from a measurement of the halo mass function, the "trivial" effect due to the filter, and to remain with the effects due to genuine non-Gaussianities. Second, putting together the corrections due to the filter with the model of a diffusing barrier, we ended up with a halo mass function which works very well in the gaussian case, see Figs. 6 and 7 of paper II, and which therefore is a meaningful starting point for the inclusion of non-Gaussian perturbations. Finally, the formalism developed in paper I can be applied, with simple modifications, to the perturbative computation of the non-Gaussian corrections. This will be the subject of the present paper.

This paper is organized as follows. In Section 2, extending to the non-Gaussian case the results presented in paper I, we show how to formulate the first-passage time problem for non-Gaussian fluctuations in terms of a path integral with boundaries, and we recall the basic points of the computation of non-markovian corrections performed in paper I. In Section 3 we compute the non-Gaussian corrections with the excursion set method, and we present our results for the halo mass function. We will see that, in the approximation in which the three-point correlator at different times is replaced by the corresponding cumulant, we recover eq. (10) exactly, including the replacement (12), except that now this replacement is not performed ad hoc to fit the N-body simulations, but is the consequence of the diffusing barrier model of paper II, which also predicts $\sqrt{a} \simeq 0.89$, in remarkable agreement with the findings of Grossi et al. (2009).[8] We will see however that this result comes out in a rather unexpected way. In fact, the "local" term that, in excursion set theory, is supposed to give back the PS result multiplied by the appropriate factor of two, actually vanishes, because for the three-point correlator (as well as for all odd-point correlators) the contribution of the image gaussian cancels the Press-Schechter contribution rather than adding up. The result (10) comes entirely from non-trivial memory terms, that have no correspondence in the naive PS approach.

We will then go beyond the approximation in which the three-point correlator at different times is replaced by the corresponding cumulant, by computing explicitly the mass function at next-to-leading order and at next-to-next-to-leading order in the small parameter σ^2/δ_c^2. We wil then find further corrections, which depends on the derivative of the correlator, and which, with respect to the small parameter σ^2/δ_c^2, are of the same order as the subleading terms given in eq. (10).

Finally, in Section 4 we present our conclusions, summarizing the findings of this series of three papers.

The focus of this paper is on the generalization of excursion set theory to non-Gaussian fluctuations. However, in appendix A we examine, with our path integral formalism, the generalization of naive PS theory to non-Gaussian fluctuations, and we will contrast it with the generalization of excursion set theory.

We have attempted to write this paper in a reasonably self-contained manner, but the reading of this paper will certainly be facilitated by a previous acquaintance with the first two papers of this series, in particular with paper I.

passage of a barrier by trajectories performing a non-markovian stochastic motion (at least order by order in the non-markovian corrections). Of course, one should not forget that excursion set theory itself gives only an approximate description of the physics involved.

[8] The parameter that we denote by a is the same as the parameter q of Grossi et al. (2009).

2. PATH INTEGRAL APPROACH TO STOCHASTIC PROBLEMS. NON-GAUSSIAN FLUCTUATIONS

2.1. *General formalism*

In this section we extend to non-Gaussian fluctuations the path integral approach that we developed in Section 3 of paper I for gaussian fluctuations. Our notation is as in paper I. In particular, we consider the density field δ smoothed over a radius R with a tophat filter in coordinate space. We denote by S the variance of the smoothed density field and, as usual in excursion set theory, we consider δ as a variable evolving stochastically with respect to the "pseudotime" S, (see e.g. Sections 2 of paper I). The statistical properties of a random variable $\delta(S)$ are specified by its connected correlators

$$\langle \delta(S_1)\ldots\delta(S_p)\rangle_c\,, \tag{13}$$

where the subscript c stands for "connected". We will also use the notation

$$\langle \delta^p(S)\rangle_c \equiv \mu_p(S)\,, \tag{14}$$

when all arguments $S_1, S_2, \ldots$ are equal. The quantities $\mu_p(S)$ are also called the cumulants. As in paper I, we consider an ensemble of trajectories all starting at $S_0 = 0$ from an initial position $\delta(0) = \delta_0$ (we will typically choose $\delta_0 = 0$ but the computation can be performed in full generality) and we follow them for a "time" S. We discretize the interval $[0,S]$ in steps $\Delta S = \epsilon$, so $S_k = k\epsilon$ with $k = 1,\ldots n$, and $S_n \equiv S$. A trajectory is then defined by the collection of values $\{\delta_1,\ldots,\delta_n\}$, such that $\delta(S_k) = \delta_k$.

The probability density in the space of trajectories is

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n) \equiv \langle \delta_D(\delta(S_1)-\delta_1)\ldots\delta_D(\delta(S_n)-\delta_n)\rangle\,, \tag{15}$$

where δ_D denotes the Dirac delta. As in paper I, our basic object will be

$$\Pi_\epsilon(\delta_0;\delta_n;S_n) \equiv \int_{-\infty}^{\delta_c} d\delta_1 \ldots \int_{-\infty}^{\delta_c} d\delta_{n-1}\, W(\delta_0;\delta_1,\ldots,\delta_{n-1},\delta_n;S_n). \tag{16}$$

The usefulness of Π_ϵ is that it allows us to compute the first-crossing rate from first principles, without the need of postulating the existence of an absorbing barrier. In fact, the quantity

$$\int_{-\infty}^{\delta_c} d\delta_n\, \Pi_\epsilon(\delta_0;\delta_n;S_n) \tag{17}$$

gives the probability that at "time" S_n a trajectory always stayed in the region $\delta < \delta_c$, for all times' smaller than S_n. The rate of change of this quantity is therefore equal to minus the rate at which trajectories cross for the first time the barrier, so the first-crossing rate is

$$\mathcal{F}(S_n) = -\frac{\partial}{\partial S_n}\int_{-\infty}^{\delta_c} d\delta_n\, \Pi_\epsilon(\delta_0;\delta_n;S_n)\,. \tag{18}$$

The halo mass function is then obtained from the first-crossing rate using eq. (1) together with (see eq. (33) of paper I)

$$f(\sigma) = 2\sigma^2\mathcal{F}(\sigma^2)\,, \tag{19}$$

where $S = \sigma^2$. For comparison, it is also useful to introduce

$$\Pi_{\rm PS,\epsilon}(\delta_0;\delta_n;S_n) \equiv \int_{-\infty}^{\infty} d\delta_1\ldots d\delta_{n-1}\, W(\delta_0;\delta_1,\ldots,\delta_{n-1},\delta_n;S_n)\,. \tag{20}$$

So, $\Pi_{\rm PS,\epsilon}(\delta_0;\delta_n;S_n)$ is the probability density of arriving in δ_n at time S_n, starting from δ_0 at time $S_0 = 0$, through any possible trajectory, while $\Pi_\epsilon(\delta_0;\delta_n;S_n)$ is the probability density of

arriving in δ_n at time S, again starting from δ_0 at time $S_0=0$, through trajectories that never exceeded δ_c. Observe that in both cases the final point δ_n ranges over $-\infty<\delta_n<\infty$. Inserting eq. (15) into eq. (20) and carrying out the integrals over $d\delta_1\ldots d\delta_{n-1}$ we see that

$$\Pi_{\mathrm{PS},\epsilon}(\delta_0;\delta_n;S_n)=\langle\delta_D(\delta(S_n)-\delta_n)\rangle\,. \tag{21}$$

Therefore $\Pi_{\mathrm{PS},\epsilon}$ can depend only on the correlators (13) with all times equal to S_n, i.e. on the cumulants $\mu_p(S_n)$. In contrast, $\Pi_\epsilon(\delta_0;\delta_n;S_n)$ is a much more complicated object, that depends on the multi-time correlators given in eq. (13).

Furthermore, we see that $\Pi_{\mathrm{PS},\epsilon}$ is actually independent of ϵ, since the integration over the intermediate positions has been carried out explicitly, and the result depend only on δ_n and S_n. Thus, we will write $\Pi_{\mathrm{PS},\epsilon}$ simply as Π_{PS}. In contrast, Π_ϵ depends on ϵ, and we keep this ϵ dependence explicit. We are finally interested in its continuum limit, $\Pi_{\epsilon=0}$, and we have already seen in paper I that taking the limit $\epsilon\to 0$ of Π_ϵ is non-trivial. So, despite their formal similarity, Π_ϵ and Π_{PS} are two very different objects. The distribution function Π_{PS} has a trivial continuum limit, and depend only on the cumulants, while Π_ϵ depends on the full correlation functions (13), and its continuum limit is non-trivial. All the complexity enters in Π_ϵ through the presence of a boundary in the integration domain, since the variables δ_i are integrated only up to δ_c.

The use of Π_{PS} generalizes to non-Gaussian fluctuations the original PS theory, since we are integrating over all trajectories, including trajectories that perform multiple up- and down-crossings of the critical value δ_c, and therefore suffers of the same cloud-in-cloud problem of the original PS theory. In the literature (Chiu et al. 1997; Robinson & Baker 2000; Matarrese et al. 2000; LoVerde et al. 2008) this density functional has then been used together with the ad hoc prescription that we must multiply the mass function derived from it by a "fudge factor" that ensures that the total mass of the universe ends up in virialized objects. For gaussian fluctuations this is the well-known factor of two of Press and Schechter, while for non-Gaussian theories it is different, although typically close to two.

In contrast, Π_ϵ generalizes to non-Gaussian fluctuations the approach of the excursion set method, where the "cloud-in-cloud" problem is cured focusing on the first-passage time of the trajectory, and no ad hoc multiplicative factor is required. So, Π_ϵ is the correct quantity to compute. From the comparison of Π_ϵ and Π_{PS} performed above, we understand that the difference between the two is not just a matter of an overall normalization factor. As we have seen above, in Π_{PS} all the information contained in the correlators at different "times" get lost, since it depends only on the cumulants. The correlators at different time contain, however, important physical information. Recalling that the role of "time" is actually played by $S(R)$, the correlators at different time are actually correlators between density fields at different smoothing scales R_1, R_2, etc., and therefore carry the information on the dependence of halo formation on the environment and on the past history. These informations are intrinsically non-markovian, which is the reason why Π_ϵ is much more difficult to compute. However, these correlations are physically very important, especially when we study the non-Gaussianities, and are completely lost in the extension of PS theory based on Π_{PS}. For this reason, our real interest is in computing the distribution function Π_ϵ, while Π_{PS} will only be considered as a benchmark against which we can compare the results provided by Π_ϵ.

The first problem that we address is how to express $\Pi_{\mathrm{PS}}(\delta_0;\delta;S)$ and $\Pi_\epsilon(\delta_0;\delta;S)$, in terms of the correlators of the theory. Using the integral representation of the Dirac delta

$$\delta_D(x)=\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,e^{-i\lambda x}\,, \tag{22}$$

we write eq. (15) as

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n)=\int_{-\infty}^{\infty}\frac{d\lambda_1}{2\pi}\ldots\frac{d\lambda_n}{2\pi}\,e^{i\sum_{i=1}^n\lambda_i\delta_i}\langle e^{-i\sum_{i=1}^n\lambda_i\delta(S_i)}\rangle\,. \tag{23}$$

We must therefore compute

$$e^Z\equiv\langle e^{-i\sum_{i=1}^n\lambda_i\delta(S_i)}\rangle\,. \tag{24}$$

This is a well-known object both in quantum field theory and in statistical mechanics, since it is the generating functional of the connected Green's functions, see e.g. Stratonovich (1967). To a field theorist this is even more clear if we define the "current" J from $-i\lambda=\epsilon J$, and we use a continuous notation, so that

$$e^Z=\langle e^{i\int dS\,J(S)\delta(S)}\rangle\,. \tag{25}$$

Therefore

$$\begin{aligned}Z&=\sum_{p=2}^{\infty}\frac{(-i)^p}{p!}\sum_{i_1=1}^{n}\ldots\sum_{i_p=1}^{n}\lambda_{i_1}\ldots\lambda_{i_p}\,\langle\delta_{i_1}\ldots\delta_{i_p}\rangle_c\\&=-\frac{1}{2}\lambda_i\lambda_j\,\langle\delta_i\delta_j\rangle_c+\frac{(-i)^3}{3!}\,\lambda_i\lambda_j\lambda_k\,\langle\delta_i\delta_j\delta_k\rangle_c\\&\quad+\frac{(-i)^4}{4!}\,\lambda_i\lambda_j\lambda_k\lambda_l\,\langle\delta_i\delta_j\delta_k\delta_l\rangle_c+\ldots\,,\end{aligned} \tag{26}$$

where $\delta_i=\delta(S_i)$ and the sum over $i,j,\ldots$ is understood. This gives

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n)=\int\mathcal{D}\lambda \tag{27}$$
$$\exp\left\{i\sum_{i=1}^n\lambda_i\delta_i+\sum_{p=2}^{\infty}\frac{(-i)^p}{p!}\sum_{i_1=1}^{n}\ldots\sum_{i_p=1}^{n}\lambda_{i_1}\ldots\lambda_{i_p}\,\langle\delta_{i_1}\ldots\delta_{i_p}\rangle_c\right\}\,,$$

where

$$\int\mathcal{D}\lambda\equiv\int_{-\infty}^{\infty}\frac{d\lambda_1}{2\pi}\ldots\frac{d\lambda_n}{2\pi}\,, \tag{28}$$

so

$$\Pi_{\mathrm{PS}}(\delta_0;\delta_n;S_n)=\int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda \tag{29}$$
$$\exp\left\{i\sum_{i=1}^n\lambda_i\delta_i+\sum_{p=2}^{\infty}\frac{(-i)^p}{p!}\sum_{i_1=1}^{n}\ldots\sum_{i_p=1}^{n}\lambda_{i_1}\ldots\lambda_{i_p}\,\langle\delta_{i_1}\ldots\delta_{i_p}\rangle_c\right\}\,,$$

and

$$\Pi_\epsilon(\delta_0;\delta_n;S_n)=\int_{-\infty}^{\delta_c}d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda \tag{30}$$
$$\exp\left\{i\sum_{i_1=1}^n\lambda_i\delta_i+\sum_{p=2}^{\infty}\frac{(-i)^p}{p!}\sum_{i=1}^{n}\ldots\sum_{i_p=1}^{n}\lambda_{i_1}\ldots\lambda_{i_p}\,\langle\delta_{i_1}\ldots\delta_{i_p}\rangle_c\right\}\,.$$

2.2. *Perturbation over the markovian case*

As it was found in the classical paper by Bond et al. (1991), when the density $\delta(R)$ is smoothed with a sharp filter in momentum space it satisfies the equation

$$\frac{\partial\delta(S)}{\partial S} = \eta(S)\,, \tag{31}$$

where here $S = \sigma^2(R)$ is the variance of the linear density field smoothed on the scale R and computed with a sharp filter in momentum space, while $\eta(S)$ satisfies

$$\langle\eta(S_1)\eta(S_2)\rangle = \delta(S_1 - S_2)\,. \tag{32}$$

Equations (31) and (32) are formally the same as a Langevin equation with a Dirac-delta noise $\eta(S)$. In this case, as discussed in paper I,

$$\langle\delta(S_i)\delta(S_j)\rangle_c = \min(S_i, S_j)\,, \tag{33}$$

and for gaussian fluctuations, where all n-point connected correlators with $n \geq 3$ vanish, the probability density W can be computed explicitly,

$$W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n) = \frac{1}{(2\pi\epsilon)^{n/2}}\,\exp\left\{-\frac{1}{2\epsilon}\sum_{i=0}^{n-1}(\delta_{i+1}-\delta_i)^2\right\}, \tag{34}$$

where the superscript "gm" (gaussian-markovian) reminds that this value of W is computed for gaussian fluctuations, whose dynamics with respect to the smoothing scale is markovian. Using this result, in paper I we have shown that, in the continuum limit, the distribution function $\Pi_{\epsilon=0}(\delta;S)$, computed with a sharp filter in momentum space, satisfies a Fokker-Planck equation with the boundary condition $\Pi_{\epsilon=0}(\delta_c,S) = 0$, and we have therefore recovered, from our path integral approach, the standard result of excursion set theory,

$$\Pi^{\rm gm}_{\epsilon=0}(\delta_0;\delta;S) = \frac{1}{\sqrt{2\pi S}}\left[e^{-(\delta-\delta_0)^2/(2S)} - e^{-(2\delta_c-\delta_0-\delta)^2/(2S)}\right]. \tag{35}$$

For a tophat filter in coordinate space, we have found in paper I that eq. (33) is replaced by

$$\langle\delta(S_i)\delta(S_j)\rangle_c = \min(S_i,S_j) + \Delta(S_i,S_j)\,, \tag{36}$$

where S is now the variance of the linear density field computed with tophat filter in coordinate space. We found that (for the ΛCDM model used in paper I) $\Delta(S_i,S_j)$ is very well approximated by the simple analytic expression

$$\Delta(S_i,S_j) \simeq \kappa\,\frac{S_i(S_j - S_i)}{S_j}\,, \tag{37}$$

where $S_i \leq S_j$ (the value for $S_i > S_j$ is obtained by symmetry, since $\Delta(S_i,S_j) = \Delta(S_j,S_i)$), and $\kappa(R)$ is given in eq. (4). The term $\min(S_i,S_j)$ in eq. (36) would be obtained if the dynamics where governed by the Langevin equation eq. (31), written with respect to the variance S computed with the tophat filter in coordinate space, and with a Dirac delta noise, and therefore describes the markovian part of the dynamics. The term $\Delta(S_i,S_j) \equiv \Delta_{ij}$ is a correction that reflects the fact that, when one uses a tophat filter in coordinate space, the underlying dynamics is non-markovian. Observe that the full two-point correlator (36) cannot be obtained from an underlying Langevin equation and, as a consequence, the probability distribution $\Pi_\epsilon(\delta_0;\delta_n;S_n)$ does not satisfy any local generalization of the Fokker-Planck equation, see the discussion below eq. (83) of paper I. However, the formalism developed in paper I allowed us to compute $\Pi_\epsilon(\delta_0;\delta_n;S_n)$ directly from its path integral representation,

$$\Pi_\epsilon(\delta_0;\delta_n;S_n) = \int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda \times\exp\left\{i\lambda_i\delta_i - \frac{1}{2}[\min(S_i,S_j)+\Delta(S_i,S_j)]\lambda_i\lambda_j\right\}, \tag{38}$$

by expanding perturbatively in $\Delta(S_i,S_j)$. The zeroth-order term simply gives eq. (35), i.e. the standard excursion set result, with the variance of the filter that we are using. The first correction is given by

$$\begin{aligned}\Pi^{\Delta 1}_\epsilon(\delta_0;\delta_n;S_n) &\equiv \int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\frac{1}{2}\sum_{i,j=1}^{n}\Delta_{ij}\partial_i\partial_j \\ &\times\int\mathcal{D}\lambda\,\exp\left\{i\sum_{i=1}^{n}\lambda_i\delta_i - \frac{1}{2}\sum_{i,j=1}^{n}\min(S_i,S_j)\lambda_i\lambda_j\right\} \\ &= \frac{1}{2}\sum_{i,j=1}^{n}\Delta_{ij}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\,,\end{aligned} \tag{39}$$

where we used the notation $\partial_i = \partial/\partial\delta_i$ anf the identity $\lambda e^{i\lambda x} = -i\partial_x e^{i\lambda x}$. This quantity has been computed explicitly in Section 5.3 of paper I, and the corresponding result for the halo mass function is given by eq. (3). In this paper we will perform a similar computation for the correction induced by the three-point function.

3. EXTENSION OF EXCURSION SET THEORY TO NON-GAUSSIAN FLUCTUATIONS

If in eq. (30) we only retain the three-point correlator, and we use the tophat filter in coordinate space, we have

$$\Pi_\epsilon(\delta_0;\delta_n;S_n) = \int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda \times\exp\left\{i\lambda_i\delta_i - \frac{1}{2}[\min(S_i,S_j)+\Delta_{ij}]\lambda_i\lambda_j + \frac{(-i)^3}{6}\langle\delta_i\delta_j\delta_k\rangle\lambda_i\lambda_j\lambda_k\right\}. \tag{40}$$

Expanding to first order, Δ_{ij} and $\langle\delta_i\delta_j\delta_k\rangle$ do not mix, so we must compute

$$\Pi^{(3)}_\epsilon(\delta_0;\delta_n;S_n) \equiv -\frac{1}{6}\sum_{i,j,k=1}^{n}\langle\delta_i\delta_j\delta_k\rangle\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j\partial_k W^{\rm gm}\,, \tag{41}$$

where the superscript (3) in $\Pi^{(3)}_\epsilon$ refers to the fact that this is the contribution linear in the three-point correlator. In principle the expression given in eq. (41) can be computed using the formalism that we developed in paper I. In the continuum limit the triple sum over i,j,k in eq. (41) becomes a triple integral over intermediate time variables dS_i, dS_j, dS_k, each one integrated from zero to S_n, so the full result is given by a triple time integral involving $\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle$, which is not very illuminating.

Fortunately, such a full computation is not necessary either. Remember in fact that the non-Gaussianities are particularly interesting at large masses. Large masses correspond to small values of the variance $S = \sigma^2(M)$. Each of the integrals over dS_i, dS_j, dS_k must therefore be performed over an interval $[0, S_n]$ that shrinks to zero as $S_n \to 0$. In this limit it is not necessary to take into account the exact functional

form of $\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle$. Rather, to lowest order we can replace it simply by $\langle\delta^3(S_n)\rangle$. More generally, we can expand the three-point correlator in a triple Taylor series around the point $S_i = S_j = S_k = S_n$. We introduce the notation

$$G_3^{(p,q,r)}(S_n) \equiv \left[\frac{d^p}{dS_i^p}\frac{d^q}{dS_j^q}\frac{d^r}{dS_k^r}\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle\right]_{S_i=S_j=S_k=S_n} . \tag{42}$$

Then

$$\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle = \sum_{p,q,r=0}^{\infty}\frac{(-1)^{p+q+r}}{p!q!r!}(S_n-S_i)^p(S_n-S_j)^q(S_n-S_k)^r G_3^{(p,q,r)}(S_n) . \tag{43}$$

We expect (and we will verify explicitly in the following) that terms with more and more derivatives give contributions to the function $f(\sigma)$, defined in eq. (1), that are subleading in the limit of small σ, i.e. for $\sigma/\delta_c \ll 1$. So, we expect that the leading contribution to the halo mass function will be given by the term in eq. (43) with $p=q=r=0$. At next-to-leading order we must also include the contribution of the terms in eq. (43) with $p+q+r=1$, i.e. the three terms ($p=1,q=0,r=0$), ($p=0,q=1,r=0$) and ($p=0,q=0,r=1$), at next-to-next-to-leading order we must include the contribution of the terms in eq. (43) with $p+q+r=2$, and so on.

Observe that, in a general theory, the functions $G_3^{(p,q,r)}(S_n)$ with different values of (p,q,r) are all independent of each other; for instance,

$$G_3^{(1,0,0)}(S_n) = \left[\frac{d}{dS_i}\langle\delta(S_i)\delta^2(S_n)\rangle\right]_{S_i=S_n} , \tag{44}$$

is in general not the same as

$$\frac{1}{3}\left[\frac{d}{dS}\langle\delta^3(S)\rangle\right]_{S=S_n} , \tag{45}$$

so $G_3^{(1,0,0)}(S_n)$ cannot be written as a derivative of $G_3^{(0,0,0)}(S_n)$. The terms $G_3^{(p,q,r)}(S_n)$ in eq. (43) must all be treated as independent functions, that characterize the most general non-gaussian theory (except, of course, for the fact that $G_3^{(p,q,r)}(S_n)$ is symmetric under exchanges of p,q,r). However, for the purpose of organizing the expansion in leading term, subleading terms, etc., we can reasonably expect that, for small S_n

$$G_3^{(p,q,r)}(S_n) \sim S_n^{-(p+q+r)}\langle\delta^3(S_n)\rangle , \tag{46}$$

i.e. each derivative $\partial/\partial S_i$, when evaluated in $S_i = S_n$, gives a factor of order $1/S_n$. This ordering will be assumed when we present our final result for the halo mass function below. However, our formalism allows us to compute each contribution separately, so our results below can be easily generalized in order to cope with a different hierarchy between the various $G_3^{(p,q,r)}(S_n)$.

3.1. *Leading term*

The leading term in $\Pi^{(3)}$ is

$$\Pi_\epsilon^{(3,\mathrm{L})}(\delta_0;\delta_n;S_n) = -\frac{\langle\delta_n^3\rangle}{6}\sum_{i,j,k=1}^{n}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\partial_i\partial_j\partial_k W^{\mathrm{gm}} , \tag{47}$$

where the superscript "L" stands for "leading". This expression can be computed very easily by making use of a trick that we already introduced in paper I. Namely, we consider the derivative of $\Pi_\epsilon^{\mathrm{gm}}$ with respect to δ_c (which, when we use the notation $\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n)$, is not written explicitly in the list of variable on which $\Pi_\epsilon^{\mathrm{gm}}$ depends, but of course enters as upper integration limit in eq. (16)). The first derivative with respect to δ_c can be written as (see eq. (B8) of paper I)

$$\frac{\partial}{\partial\delta_c}\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n) = \sum_{i=1}^{n-1}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\partial_i W^{\mathrm{gm}} , \tag{48}$$

since, when $\partial/\partial\delta_c$ acts on the upper integration limit of the integral over $d\delta_i$, it produces $W(\delta_1,\ldots,\delta_i=\delta_c,\ldots,\delta_n;S_n)$, which is the same as the integral of $\partial_i W$ with respect to $d\delta_i$ from $\delta_i=-\infty$ to $\delta_i=\delta_c$. Similarly

$$\frac{\partial^2}{\partial\delta_c^2}\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n) = \sum_{i,j=1}^{n-1}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j W^{\mathrm{gm}} , \tag{49}$$

see eqs. (B9) and (B10) of paper I. In the same way we find that

$$\frac{\partial^3}{\partial\delta_c^3}\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n) = \sum_{i,j,k=1}^{n-1}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j\partial_k W^{\mathrm{gm}} . \tag{50}$$

The right-hand side of this identity is not yet equal to the quantity that appears in eq. (47), since there the sums run up to n while in eq. (50) they only run up to $n-1$. However, what we need is not really $\Pi_\epsilon^{(3)}(\delta_0;\delta_n;S_n)$, but rather its integral over $d\delta_n$, which is the quantity that enters in eq. (18). Then we consider

$$\int_{-\infty}^{\delta_c} d\delta_n\,\Pi_\epsilon^{(3,\mathrm{L})}(\delta_0;\delta_n;S_n) = -\frac{1}{6}\langle\delta_n^3\rangle \times\sum_{i,j,k=1}^{n}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\partial_i\partial_j\partial_k W^{\mathrm{gm}} , \tag{51}$$

and we can now use the identity

$$\begin{aligned}&\sum_{i,j,k=1}^{n}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\,\partial_i\partial_j\partial_k W^{\mathrm{gm}}\\ &= \frac{\partial^3}{\partial\delta_c^3}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\,W^{\mathrm{gm}}\\ &= \frac{\partial^3}{\partial\delta_c^3}\int_{-\infty}^{\delta_c} d\delta_n\;\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n) ,\end{aligned} \tag{52}$$

so

$$\int_{-\infty}^{\delta_c} d\delta_n\,\Pi_\epsilon^{(3,\mathrm{L})}(\delta_0;\delta_n;S_n) = -\frac{\langle\delta_n^3\rangle}{6}\frac{\partial^3}{\partial\delta_c^3}\int_{-\infty}^{\delta_c} d\delta_n\,\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_n;S_n) . \tag{53}$$

From eq. (35), setting for simplicity $\delta_0 = 0$,

$$\Pi_{\epsilon=0}^{\mathrm{gm}}(\delta_0=0;\delta_n;S_n) = \frac{1}{\sqrt{2\pi S_n}}\left[e^{-\delta_n^2/(2S_n)} - e^{-(2\delta_c-\delta_n)^2/(2S_n)}\right] . \tag{54}$$

Inserting this into eq. (53) we immediately find the result in the continuum limit,

$$\int_{-\infty}^{\delta_c} d\delta_n\,\Pi_{\epsilon=0}^{(3,\mathrm{L})}(0;\delta_n;S_n) = \frac{\langle\delta_n^3\rangle}{3\sqrt{2\pi}\,S_n^{3/2}}\left(1-\frac{\delta_c^2}{S_n}\right)e^{-\delta_c^2/(2S_n)} . \tag{55}$$

We now insert this result into eqs. (18) and (19) and we express the result in terms of the normalized skewness

$$\mathcal{S}_3(\sigma) \equiv \frac{1}{S^2}\langle\delta^3(S)\rangle\,. \tag{56}$$

Putting the contribution of $\Pi^{(3,\mathrm{L})}$ together with the gaussian contribution, we find

$$f(\sigma) = \left(\frac{2}{\pi}\right)^{1/2}\,\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)} \tag{57}$$
$$\times\left\{1+\frac{\sigma^2}{6\delta_c}\left[\mathcal{S}_3(\sigma)\left(\frac{\delta_c^4}{\sigma^4}-\frac{2\delta_c^2}{\sigma^2}-1\right)+\frac{d\mathcal{S}_3}{d\ln\sigma}\,\left(\frac{\delta_c^2}{\sigma^2}-1\right)\right]\right\}.$$

Remarkably, this agrees exactly with the result obtained by LoVerde et al. (2008), performing an Edgeworth expansion of the non-Gaussian generalization of Press-Schechter theory, see eq. (10).

However, the fact that a naive non-Gaussian generalization of PS theory gives the same result that we have obtained from the non-Gaussian generalization of excursion set theory (at least to leading order for small σ/δ_c; we will see below that the subleading term gets corrections) is somewhat accidental, as can be realized as follows. In the sum over i,j,k of $\partial_i\partial_j\partial_k$ in eq. (47), it is useful to separate the contribution with $i=j=k=n$ from the rest. Recall that in PS theory the upper integration limit for the variables $d\delta_1,\ldots,d\delta_{n-1}$ is $+\infty$ rather than δ_c (which reflects the fact that in PS theory one looks at the probability that, at a given smoothing radius, the smoothed density is above threshold, regardless of whether it was already above threshold for some larger smoothing radius). If in eq. (47) we replaced the upper integration limit δ_c with $+\infty$, a derivative ∂_i with $i<n$ would integrate by parts to zero. The terms where at least one of the indices i,j or k is strictly smaller than n therefore have no counterpart in PS theory. The term where all indices $i,j,\ldots$ are equal to n, in contrast, are local terms, which depends only on the cumulants rather than on the correlators at different points, and that can have a correspondence with PS theory. In the gaussian case, it is just such a local term that gives back the PS result, together with the factor of two that in PS theory was added by hand. Formally, this comes from the fact that, in the gaussian case, the excursion set probability distribution is the difference between the original PS gaussian and an "image" gaussian, and these two terms give contributions that add up when computing the first crossing rate.

In the case of the three-point correlator the situation is however different. Denoting by $\Pi^{(3,\mathrm{La})}$ the contribution to $\Pi^{(3,\mathrm{L})}$ obtained by setting $i=j=k=n$ in eq. (47), we have

$$\Pi^{(3,\mathrm{La})}_{\epsilon=0}(0;\delta_n;S_n) = -\frac{1}{6}\langle\delta_n^3\rangle\partial_n^3\Pi^{\mathrm{gm}}_{\epsilon=0}(0;\delta_n;S_n)\,, \tag{58}$$

and therefore

$$\int_{-\infty}^{\delta_c} d\delta_n\,\Pi^{(3,\mathrm{La})}_{\epsilon=0}(0;\delta_n;S_n) = -\frac{1}{6}\langle\delta_n^3\rangle\left[\partial_n^2\Pi^{\mathrm{gm}}_{\epsilon=0}(0;\delta_n;S_n)\right]_{\delta_n=\delta_c} = 0\,. \tag{59}$$

This result is in a sense surprising. Since PS theory gives a wrong normalization factor, missing a factor of two in the gaussian case, and a factor close to two in the non-Gaussian case, what is done in the literature when one uses PS theory is to take the PS result and multiply it by hand by a factor of two (or, for non-Gaussian fluctuations, close to two), assuming that this would come out from a proper treatment of the cloud-in-cloud problem, i.e. from excursion set theory. We see however that this is not at all the case. In excursion set theory $\Pi^{\mathrm{gm}}_{\epsilon=0}$ is a difference of two gaussians, see eq. (35), so all its derivative with respect to δ_n of odd order, evaluated in $\delta_n=\delta_c$ are twice as large as for a single gaussian, but the function itself, as well as all its derivative with respect to δ_n of even order, evaluated in $\delta_n=\delta_c$, are zero, i.e. the contribution from the second gaussian cancels the first contribution, rather than adding up. Since in eq. (59) appears the second derivative of $\Pi^{\mathrm{gm}}_{\epsilon=0}$ in $\delta_n=\delta_c$, this term vanishes. We therefore see that the logic behind the use of PS theory for non-Gaussian fluctuations, namely (1): compute with a naive extension of PS theory to non-Gaussian fluctuations and (2): multiply the result by hand by a "fudge factor" $\simeq 2$, assuming that it would come out from a solution of the cloud-in-cloud problem, is not justified. For the contribution linear in the three-point correlator $\langle\delta_n^3\rangle$, this "fudge factor" is actually zero, and the result comes entirely from terms with at least one derivative ∂_i with $i<n$, which have no counterpart in a non-Gaussian extension of PS theory. Above we have computed the excursion set theory result performing at once the sum over i,j,k, using the trick given in eq. (52). In appendix B we compute separately the terms in the sum over i,j,k with one or more indices equal to n, and we check that they give back eq. (55).

In Sections 3.2 and 3.3 we will compute the corrections to eq. (57) to next-to-leading and to next-to-next-to-leading order. We also need to take into account that the barrier must be treated as diffusing, see paper II, and we must include the corrections due to the tophat filter in coordinate space. This will be done in Section 3.4.

Before leaving this section we observe that, in the approximation in which the correlators are replaced by the cumulants, the effects of the higher-order correlators can also be computed very simply. For instance, the effect of the four-point function $\langle\delta_n^4\rangle$ is obtained using

$$\sum_{i,j,k,l=1}^{n}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\,\partial_i\partial_j\partial_k\partial_l W^{\mathrm{gm}}$$
$$=\frac{\partial^4}{\partial\delta_c^4}\int_{-\infty}^{\delta_c} d\delta_n\;\Pi^{\mathrm{gm}}_{\epsilon}(\delta_0;\delta_n;S)\,. \tag{60}$$

3.2. *The next-to-leading term*

Using eqs. (41) and (43), at next-to-leading order we get

$$\int_{-\infty}^{\delta_c} d\delta_n\,\Pi^{(3,\mathrm{NL})}_{\epsilon}(\delta_0;\delta_n;S_n) = \frac{1}{2}\,G_3^{(1,0,0)}(S_n) \tag{61}$$
$$\times\sum_{i=1}^{n}(S_n-S_i)\sum_{j,k=1}^{n}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\partial_i\partial_j\partial_k W^{\mathrm{gm}}\,,$$

where the superscript "NL" in $\Pi^{(3,\mathrm{NL})}_{\epsilon}$ stands for next-to-leading, and we used the fact that the three terms $(p=1,q=0,r=0)$, $(p=0,q=1,r=0)$ and $(p=0,q=0,r=1)$ give the same contribution. We now use the same trick as before to eliminate $\sum_{j,k=1}^{n}\partial_j\partial_k$ in favor of $\partial^2/\partial\delta_c^2$,

$$\int_{-\infty}^{\delta_c} d\delta_n\Pi^{(3,\mathrm{NL})}_{\epsilon}(\delta_0;\delta_n;S_n) = \frac{1}{2}\,G_3^{(1,0,0)}(S_n)$$
$$\times\sum_{i=1}^{n}(S_n-S_i)\frac{\partial^2}{\partial\delta_c^2}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\partial_i W^{\mathrm{gm}}\,.$$

The remaining path integral can be computed using the technique developed in paper I, namely we write

$$\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}d\delta_n\,\partial_i W^{\mathrm{gm}} \tag{62}$$

$$= \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \, W(\delta_0;\delta_1,\ldots,\delta_i=\delta_c,\ldots,\delta_n;S_n)\,,$$

and we use

$$W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},\delta_c,\delta_{i+1},\ldots,\delta_n;S_n) \tag{63}$$
$$= W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},\delta_c;S_i) W^{\rm gm}(\delta_c;\delta_{i+1},\ldots,\delta_n;S_n-S_i)\,,$$

so

$$\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{i-1} \int_{-\infty}^{\delta_c} d\delta_{i+1} \ldots d\delta_{n-1} d\delta_n$$
$$\times W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},\delta_c;S_i) W^{\rm gm}(\delta_c;\delta_{i+1},\ldots,\delta_n;S_n-S_i)$$
$$= \Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S_i) \int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S_n-S_i)\,. \tag{64}$$

Recalling from paper I that

$$\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S) = \sqrt{\epsilon}\,\frac{1}{\sqrt{\pi}}\,\frac{\delta_c-\delta_0}{S^{3/2}} e^{-(\delta_c-\delta_0)^2/(2S)} + \mathcal{O}(\epsilon) \tag{65}$$

and

$$\Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S) = \sqrt{\epsilon}\,\frac{1}{\sqrt{\pi}}\,\frac{\delta_c-\delta_n}{S^{3/2}} e^{-(\delta_c-\delta_n)^2/(2S)} + \mathcal{O}(\epsilon)\,, \tag{66}$$

we see that the factors $\sqrt{\epsilon}$ in $\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S)$ and in $\Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S)$ combine with $\sum_i$ to produce an integral over dS_i, and

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NL})}(\delta_0;\delta_n;S_n) = \frac{1}{2\pi}\, G_3^{(1,0,0)}(S_n)$$
$$\times \int_0^{S_n} dS_i \, \frac{1}{S_i^{3/2}(S_n-S_i)^{1/2}} \tag{67}$$
$$\times \frac{\partial^2}{\partial\delta_c^2} \left[\delta_c e^{-\delta_c^2/(2S_i)} \int_{-\infty}^{\delta_c} d\delta_n \, (\delta_c-\delta_n) \exp\left\{ -\frac{(\delta_c-\delta_n)^2}{2(S_n-S_i)} \right\} \right].$$

The integral over $d\delta_n$ is easily performed writing

$$(\delta_c-\delta_n) \exp\left\{ -\frac{(\delta_c-\delta_n)^2}{2(S_n-S_i)} \right\} = (S_n-S_i)\partial_n \exp\left\{ -\frac{(\delta_c-\delta_n)^2}{2(S_n-S_i)} \right\}, \tag{68}$$

so it just gives (S_n-S_i). Carrying out the second derivative with respect to δ_c and the remaining elementary integral over dS_i we get

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NL})}(\delta_0;\delta_n;S_n) = \frac{1}{\sqrt{2\pi}}\, \frac{G_3^{(1,0,0)}(S_n)}{S_n^{1/2}}\, e^{-\delta_c^2/(2S_n)}\,. \tag{69}$$

We now define

$$\mathcal{U}_3(\sigma) \equiv \frac{3G_3^{(1,0,0)}(S)}{S}\,, \tag{70}$$

where as usual $S=\sigma^2$. When the ordering given in eq. (46) holds, $\mathcal{U}_3(\sigma)$ is of the same order as the normalized skewness $\mathcal{S}_3(\sigma)$ given in eq. (56). Computing the contribution to $f(\sigma)$ from eq. (69) and putting it together with eq. (57) we finally find

$$f(\sigma) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\delta_c}{\sigma}\, e^{-\delta_c^2/(2\sigma^2)} \left[1 + \frac{\sigma^2}{6\delta_c} h_{\rm NG}(\sigma) \right], \tag{71}$$

where

$$h_{\rm NG}(\sigma) = \frac{\delta_c^4}{\sigma^4}\mathcal{S}_3(\sigma) - \frac{\delta_c^2}{\sigma^2}\left(2\mathcal{S}_3(\sigma) + \mathcal{U}_3(\sigma) - \frac{d\mathcal{S}_3}{d\ln\sigma} \right)$$
$$- \left(\mathcal{S}_3(\sigma) + \mathcal{U}_3(\sigma) + \frac{d\mathcal{S}_3}{d\ln\sigma} + \frac{d\mathcal{U}_3}{d\ln\sigma} \right). \tag{72}$$

We have ordered the terms in $h_{\rm NG}(\sigma)$ according to their importance in the limit of small σ/δ_c assuming, according to eq. (46), that $\mathcal{U}_3(\sigma)$ is of the same order as $\mathcal{S}_3(\sigma)$. The leading order is given by $(\delta_c/\sigma)^4\mathcal{S}_3(\sigma)$ and, as we have seen, it comes only from $\Pi^{(3,{\rm L})}$. The next-to-leading order in $h_{\rm NG}(\sigma)$ is given by the terms proportional to $(\delta_c/\sigma)^2$, and we see that it is affected by the terms with $p+q+r=1$ in the expansion of eq. (43). The terms in $h_{\rm NG}(\sigma)$ which are $\mathcal{O}(1)$ with respect to the large parameter δ_c/σ are next-to-next-to-leading order corrections and, if we wish to include them, we must for consistency include also the contribution from the terms with $p+q+r=2$ in the expansion of eq. (43). We compute them in the next subsection.

Observe also that typically $\mathcal{S}_3$ depends very weakly on the smoothing scale R and hence on σ. For instance, in $f_{\rm NL}$-theories it changes only by a factor $\simeq 3$ as R is changed by a factor 100, from $0.1\,{\rm Mpc}/h$ to $10\,{\rm Mpc}/h$, see Matarrese et al. (2000). Therefore, even if parametrically $d\mathcal{S}_3/d\ln\sigma$ has the same power-law behavior as $\mathcal{S}_3$, its prefactor will typically be numerically small.

3.3. *The next-to-next-to-leading term*

Using eqs. (41) and (43) and keeping the terms with $p+q+r=2$ we find two kind of contributions. The first has ($p=2, q=r=0$), with a combinatorial factor of three and the second has ($p=q=1, r=0$), again with a combinatorial factor of three. We denote the contribution to $\Pi^{(3)}$ at next-to-next-to-leading (NNL) order by $\Pi^{(3,{\rm NNL})}$, and the two separate contribution with ($p=2, q=r=0$) and with ($p=q=1, r=0$) as $\Pi^{(3,{\rm NNLa})}$ and $\Pi^{(3,{\rm NNLb})}$, respectively. Thus,

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NNLa})}(\delta_0;\delta_n;S_n) = -\frac{1}{4}\, G_3^{(2,0,0)}(S_n) \tag{73}$$
$$\times \sum_{i=1}^{n} (S_n-S_i)^2 \sum_{j,k=1}^{n} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_i\partial_j\partial_k W^{\rm gm}\,,$$

and

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NNLb})}(\delta_0;\delta_n;S_n) = -\frac{1}{2}\, G_3^{(1,1,0)}(S_n) \tag{74}$$
$$\times \sum_{i,j=1}^{n} (S_n-S_i)(S_n-S_j) \sum_{k=1}^{n} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_i\partial_j\partial_k W^{\rm gm}\,.$$

The first term is straightforward to compute. We use again the trick of eliminating $\sum_{j,k=1}^n \partial_j\partial_k$ in favor of $\partial^2/\partial\delta_c^2$, and we proceed just as in Section 3.2. The result is

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NNLa})}(\delta_0;\delta_n;S_n) = -\frac{3}{4\pi}\, G_3^{(2,0,0)}(S_n)$$
$$\times \left[\sqrt{2\pi}\, S_n^{1/2} e^{-\delta_c^2/(2S_n)} - \pi\delta_c \,{\rm Erfc}\left(\frac{\delta_c}{\sqrt{2S_n}} \right) \right], \tag{75}$$

where Erfc is the complementary error function.

The computation of eq. (74) is more complicated, but can be performed with the formalism that we have developed in paper I, see in particular appendix B of paper I. The factor $\sum_{k=1}^n \partial_k$ is eliminated as usual in favor of $\partial/\partial\delta_c$. We also observe that, in eq. (74), the terms with $i=n$ or $j=n$ do not contribute, because of the factor $(S_n-S_i)(S_n-S_j)$, and we separate the sum over i,j into the term with $i=j$ and twice the term with $i<j$. The first term is

$$I_1 \equiv -\frac{1}{2}\, G_3^{(1,1,0)}(S_n) \sum_{i=1}^{n-1} (S_n-S_i)^2$$

$$\times \frac{\partial}{\partial \delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_i^2 W^{\rm gm}\,, \qquad (76)$$

and the second is

$$I_2 \equiv -G_3^{(1,1,0)}(S_n) \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} (S_n - S_i)(S_n - S_j)$$
$$\times \frac{\partial}{\partial \delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_i \partial_j W^{\rm gm}\,. \qquad (77)$$

As we discussed in detail in paper I, quantities such as the right-hand side of eq. (74) are finite in the continuum limit $\epsilon \to 0$, as it is obvious physically, and as we checked explicitly in solvable examples in paper I. However, when we split the sum over the indices i, j into two separate parts, such as those given in eqs. (76) and (77), these are separately divergent in the continuum limit, and the divergence cancels when we sum them up. It is therefore necessary to regularize them carefully, and separate them into a divergent part and the finite part. Since we know that the divergent terms must cancel, we can simply extract from each term the finite part, disregarding the divergences. This is the finite part prescription discussed and tested in detail in paper I. We will denote by $\mathcal{FP}$ this procedure of extracting the finite part from terms such as (76) and (77).

The computation of the finite part of (76) is basically identical to the one that we already performed in appendix B of paper I, see in particular eqs. (B13)–(B15) and (B29) there, and the result is that this term diverges as $1/\sqrt{\epsilon}$ with no finite part, so

$$\mathcal{FP} \sum_{i=1}^{n-1} (S_n - S_i)^2 \frac{\partial}{\partial \delta_c} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_i^2 W^{\rm gm} = 0\,. \quad (78)$$

The computation (77) is also completely analogous to the computation of the "memory-of-memory" term performed in appendix B of paper I, see in particular eqs. (B17)–(B28) and (B30) there, except that we now have a factor $(S_n - S_i)(S_n - S_j)$ in the integrals over dS_i and dS_j. We can then repeat basically the same steps as detailed in appendix B of paper I, and we find

$$\mathcal{FP}[I_2] = -\frac{G_3^{(1,1,0)}(S_n)}{\pi\sqrt{2\pi}} \mathcal{FP} \int_0^{S_n} dS_i \, \frac{(S_n - S_i)}{S_i^{3/2}} \left(1 - \frac{\delta_c^2}{S_i}\right) e^{-\delta_c^2/2S_i}$$
$$\times \int_{S_i}^{S_n} dS_j \, \frac{(S_n - S_j)^{1/2}}{(S_j - S_i)^{3/2}} \exp\left\{-\frac{a^2}{2(S_j - S_i)}\right\}, \qquad (79)$$

where $a = \sqrt{\alpha\epsilon}$ and α is a numerical constant which appears when the sum over j is replaced by an integral over dS_j, see eqs. (B20)-(B24) of paper I. The integral over dS_j can be computed writing $t_n = S_n - S_i$, $t_j = S_j - S_i$, and using the identity

$$\int_0^{t_n} dt_j \, \frac{(t_n - t_j)^{1/2}}{t_j^{1/2}} \exp\left\{-\frac{a^2}{2t_j}\right\}$$
$$= \frac{\sqrt{2\pi}}{2a} \left[2t_n^{1/2} e^{-a^2/(2t_n)} - a\sqrt{2\pi}\, {\rm Erfc}\left(\frac{a}{\sqrt{2t_n}}\right)\right], \qquad (80)$$

which is proved in the same way as eqs. (115) and (116) of paper I. In this equation $a = \sqrt{\alpha\epsilon}$ goes to zero in the continuum limit. In the limit $a \to 0$ the above result displays a term divergent as $1/a$, i.e. as $1/\sqrt{\epsilon}$, which must cancel the divergence coming from (76), plus a term which is finite as $a \to 0$, which can be extracted from eq. (80) recalling that ${\rm Erfc}(0) = 1$, so

$$\mathcal{FP} \int_{S_i}^{S_n} dS_j \, \frac{(S_n - S_j)^{1/2}}{(S_j - S_i)^{3/2}} \exp\left\{-\frac{a^2}{2(S_j - S_i)}\right\} = -\pi\,. \qquad (81)$$

Computing the remaining integral over dS_i, which is finite and elementary, we find

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NNLb})}(\delta_0; \delta_n; S_n) = -\frac{2}{\pi} G_3^{(1,1,0)}(S_n)$$
$$\times \left[\sqrt{2\pi} S_n^{1/2} e^{-\delta_c^2/(2S_n)} - \pi\delta_c \, {\rm Erfc}\left(\frac{\delta_c}{\sqrt{2S_n}}\right)\right]. \qquad (82)$$

Putting together this result and eq. (75) we end up with

$$\int_{-\infty}^{\delta_c} d\delta_n \, \Pi_\epsilon^{(3,{\rm NNL})}(\delta_0; \delta_n; S_n)$$
$$= -\frac{1}{2\pi} \left(\frac{3}{2} G_3^{(2,0,0)}(S_n) + 4G_3^{(1,1,0)}(S_n)\right)$$
$$\times \left[\sqrt{2\pi} S_n^{1/2} e^{-\delta_c^2/(2S_n)} - \pi\delta_c \, {\rm Erfc}\left(\frac{\delta_c}{\sqrt{2S_n}}\right)\right]. \qquad (83)$$

We now introduce the function

$$\mathcal{V}_3(\sigma) \equiv \frac{9}{2} G_3^{(2,0,0)}(S) + 12 G_3^{(1,1,0)}(S)\,. \qquad (84)$$

According to eq. (46), $\mathcal{V}_3(\sigma)$ is parametrically of the same order as $\mathcal{S}_3(\sigma)$ and $\mathcal{U}_3(\sigma)$, as $\sigma \to 0$. We can now compute the contribution of this term to the function $h_{\rm NG}(\sigma)$ using eqs. (18), (19) and (71). Retaining only the terms that contribute up to $\mathcal{O}(1)$ in δ_c/σ, we find that the function $h_{\rm NG}(\sigma)$ is modified to

$$h_{\rm NG}(\sigma) = \frac{\delta_c^4}{\sigma^4} \mathcal{S}_3(\sigma) - \frac{\delta_c^2}{\sigma^2} \left(2\mathcal{S}_3(\sigma) + \mathcal{U}_3(\sigma) - \frac{d\mathcal{S}_3}{d\ln\sigma}\right)$$
$$- \left(\mathcal{S}_3(\sigma) + \mathcal{U}_3(\sigma) + \mathcal{V}_3(\sigma) + \frac{d\mathcal{S}_3}{d\ln\sigma} + \frac{d\mathcal{U}_3}{d\ln\sigma}\right)$$
$$+ \mathcal{O}\left(\frac{\sigma^2}{\delta_c^2}\right). \qquad (85)$$

This is the complete result for the halo mass function, up to NNL order in the small parameter σ^2/δ_c^2.

3.4. *The effects of the diffusing barrier and of the filter*

Until now we have worked with a barrier with a fixed height δ_c and we neglected the corrections due to the filter. We now include the modifications due to the fact that the height of the barrier diffuses stochastically, as discussed in paper II, and also the corrections due to the filter.

To compute the non-Gaussian term proportional to the three-point correlator with the diffusing barrier we recall, from paper II, that the first-passage time problem of a particle obeying a diffusion equation with diffusion coefficient $D = 1$, in the presence of a barrier that moves stochastically with diffusion coefficient D_B, can be mapped into the first-passage time problem of a particle with effective diffusion coefficient $(1 + D_B)$, and fixed barrier. This can be reabsorbed into a rescaling of the "time" variable $S \to (1 + D_B)S = S/a$, and therefore $\sigma \to \sigma/\sqrt{a}$. At the same time the three-point correlator must be rescaled according to $\langle\delta_n^3\rangle \to a^{-3/2}\langle\delta_n^3\rangle$ since, dimensionally, $\langle\delta_n^3\rangle$ is the same as $S^{3/2}$ (if we perform dimensional analysis as discussed below eq. (A10) of paper I), which means that $\mathcal{S}_3 \to a^{1/2}\mathcal{S}_3$, and similarly for the functions

$\mathcal{U}_3$ and $\mathcal{V}_3$.[9] Then eqs. (71) and (85) become

$$f(\sigma) = \left(\frac{2}{\pi}\right)^{1/2} \frac{a^{1/2}\delta_c}{\sigma}\, e^{-a\delta_c^2/(2\sigma^2)} \left[1+\frac{\sigma^2}{6a^{1/2}\delta_c} h_{\rm NG}(\sigma)\right], \tag{86}$$

where

$$h_{\rm NG}(\sigma) = \frac{a^2\delta_c^4}{\sigma^4}\mathcal{S}_3(\sigma) - \frac{a\delta_c^2}{\sigma^2}\left(2\mathcal{S}_3(\sigma)+\mathcal{U}_3(\sigma)-\frac{d\mathcal{S}_3}{d\ln\sigma}\right) - \left(\mathcal{S}_3(\sigma)+\mathcal{U}_3(\sigma)+\mathcal{V}_3(\sigma)+\frac{d\mathcal{S}_3}{d\ln\sigma}+\frac{d\mathcal{U}_3}{d\ln\sigma}\right) + \mathcal{O}\left(\frac{\sigma^2}{\delta_c^2}\right). \tag{87}$$

We see that the terms depending on the skewness $\mathcal{S}_3(\sigma)$ and its derivative coincide with those given in eq. (10), if we identify $\delta_{\rm eff}$ with $a^{1/2}\delta_c$. Observe, from eq. (7), that our prediction $a^{1/2} \simeq 0.89$ is in remarkable agreement with the value $a^{1/2} \simeq 0.86$ proposed by Grossi et al. (2009) from the fit to the N-body simulations (see however footnote 6).

We have therefore derived, from a first principle computation, eq. (10), which was proposed in LoVerde et al. (2008) and in Grossi et al. (2009) using a mixture of heuristic theoretical arguments (the use of a non-Gaussian extension of PS theory, rather than of the excursion set theory) and a calibration of parameters from the fit to the data of the N-body simulations (the replacement $\delta_c \to 0.86\delta_c$), and we have improved it including the effect of the functions $\mathcal{U}_3(\sigma)$ and $\mathcal{V}_3(\sigma)$, which are absent in LoVerde et al. (2008) and cannot be obtained from any naive extension of PS theory, which from the beginning contains only the cumulants, rather than the full correlation functions at different smoothing radii.

The term in eq. (87) which is dominant for small σ is the same as that of both eqs. (8) and (10), and appears to fit well the data of the N-body simulations (Grossi et al. 2009). Given the size of the error bars of the non-Gaussian N-body simulations (see e.g. Fig. 6 and 7 of Grossi et al. (2009)), it is probably difficult for the moment to test the subleading terms in eq. (87), and in particular to see the effect of the functions $\mathcal{U}_3(\sigma)$ and $\mathcal{V}_3(\sigma)$.

As a final ingredient, we must add the effect of the tophat filter function in coordinate space. When the non-gaussianities are not present, these are given by eq. (5). More generally, even the non-Gaussian corrections must be computed using the propagator $[\min(S_i,S_j)+\Delta_{ij}]$ in eq. (40), so we will apply the same correction factor found for the gaussian part also to the non-Gaussian term, and we end up with

$$f(\sigma) = (1-\tilde{\kappa})\left(\frac{2}{\pi}\right)^{1/2} \frac{a^{1/2}\delta_c}{\sigma}\, e^{-a\delta_c^2/(2\sigma^2)} \left[1+\frac{\sigma^2}{6a^{1/2}\delta_c} h_{\rm NG}(\sigma)\right] + \frac{\tilde{\kappa}}{\sqrt{2\pi}}\frac{a^{1/2}\delta_c}{\sigma}\,\Gamma\left(0,\frac{a\delta_c^2}{2\sigma^2}\right), \tag{88}$$

with $h_{\rm NG}(\sigma)$ still given by eq. (87). More generally, also the term proportional to the incomplete Gamma function could get non-Gaussian corrections, which in principle can be computed evaluating perturbatively a "mixed" term proportional to

$$\Delta_{ij}\langle\delta_k\delta_l\delta_m\rangle\partial_i\partial_j\partial_k\partial_l\partial_m \tag{89}$$

in eq. (40). However we saw in paper I that in the large mass limit, where the non-Gaussianities are important, the term proportional to the incomplete Gamma function is subleading, so we will neglect the non-Gaussian corrections to this subleading term.[10]

The relative weight of the correction due to the filter proportional to the incomplete Gamma function, and of the non-Gaussian corrections depends on the value of σ and, of course, on the value of the three-point correlator, i.e. of $\mathcal{S}_3$. In $f_{\rm NL}$ theory $\mathcal{S}_3$ increases very weakly with the mass, i.e. as $\sigma \to 0$. In the low-σ (i.e. large mass) limit we can use the asymptotic expansion of the incomplete Gamma function for large z, $\Gamma(0,z) \simeq z^{-1}e^{-z}$, and we see that, asymptotically, the term in the second line of eq. (88) depends on σ as $\sigma\exp\{-a\delta_c^2/(2\sigma^2)\}$, and therefore is small compared to both the leading and next-to-leading term in the non-Gaussian corrections, which overall behaves as $\sigma^{-3}\mathcal{S}_3(\sigma)\exp\{-a\delta_c^2/(2\sigma^2)\}$ and $\sigma^{-1}\mathcal{S}_3(\sigma)\exp\{-a\delta_c^2/(2\sigma^2)\}$, respectively. When this asymptotic behavior sets in depends, of course, on the numerical value of $\mathcal{S}_3$ so, in $f_{\rm NL}$-theory, on the value of the $f_{\rm NL}$ parameter. In any case, given a measure of $f(\sigma)$, either from galaxy surveys or from N-body simulations with non-Gaussian initial conditions, the prediction (88) allows us to disentangle the effects due to the filter from the physically interesting effects due to primordial non-Gaussianities.

4. CONCLUSIONS

To conclude this series of three papers, we summarize the main results that we obtained and, at the price of some repetition, we collect here the most important formulas that are scattered in the text. Our aim was to compute the halo mass function, i.e. the number density $n(M)dM$ of dark matter halos with mass between M and $M+dM$, both for gaussian and non-Gaussian primordial density fluctuations. This can be written as

$$\frac{dn(M)}{dM} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}(M)}{d\ln M}, \tag{90}$$

and the issue is to compute the function $f(\sigma)$. Our final result can be written as

$$f(\sigma) = (1-\tilde{\kappa})\left(\frac{2}{\pi}\right)^{1/2} \frac{a^{1/2}\delta_c}{\sigma}\, e^{-a\delta_c^2/(2\sigma^2)} \left[1+\frac{\sigma^2}{6a^{1/2}\delta_c} h_{\rm NG}(\sigma)\right] + \frac{\tilde{\kappa}}{\sqrt{2\pi}}\frac{a^{1/2}\delta_c}{\sigma}\,\Gamma\left(0,\frac{a\delta_c^2}{2\sigma^2}\right), \tag{91}$$

where $\Gamma(0,z)$ is the incomplete Gamma function. Three distinct physical effects are taken into account in this result.

One is the fact that we have treated the threshold for gravitational collapse as a stochastic variable that fluctuates around an average value, which is $\delta_c \simeq 1.686$ for the spherical collapse model, and is a rising function of σ for the ellipsoidal collapse model. As discussed in paper II, this is a way of taking into account, at least at an effective level, part of the complexity of a realistic process of halo formation, which is missed in the simple spherical or ellipsoidal collapse model.

[9] In principle we should also shift the argument σ of $\mathcal{S}_3, \mathcal{U}_3$ and $\mathcal{V}_3$. However, $\mathcal{S}_3$ depends very weakly on the smoothing scale R and hence on σ. For instance, in $f_{\rm NL}$-theories it changes only by a factor $\simeq 3$ as R is changed by a factor 100, from $0.1\,{\rm Mpc}/h$ to $10\,{\rm Mpc}/h$, see Matarrese et al. (2000). In most situation, we can then neglect the rescaling of the argument of $\mathcal{S}_3$, and we expect that the same holds for $\mathcal{U}_3$ and $\mathcal{V}_3$.

[10] Furthermore, one must be aware of the fact that the term proportional to $h_{\rm NG}(\sigma)$ might in general receive corrections from the tophat filter that have not exactly the same form as that of the gaussian term. Again, in principle these can be obtained by computing the term proportional to $\Delta_{ij}\langle\delta_k\delta_l\delta_m\rangle\partial_i\partial_j\partial_k\partial_l\partial_m$ in the expansion of the path integral.

Furthermore, the stochasticity of the barrier reflects uncertainties in the operative definition of what is a dark matter halo. The inclusion of a diffusing barrier gives rise to the constant a in the above result. This constant enters also in the exponential, thereby modifying dramatically the behavior predicted by PS theory. Our prediction is $a \simeq 0.80$, i.e. $\sqrt{a} \simeq 0.89$, which gives a remarkable agreement with the data from N-body simulations. For instance Grossi et al. (2009), from the fit to the N-body simulation, find $\sqrt{a} \simeq 0.86$.

A second effect included in eq. (91) is that we have properly accounted for the fact that the comparison with the data, whether observational or from N-body simulations, requires the use of a tophat filter function in coordinate space. In the classical paper of Bond et al. (1991), using a tophat filter in *momentum* space, the computation of $f(\sigma)$ was reduced to a first-passage time problem for a quantity that obeys a Langevin equation, and therefore the underlying dynamics is markovian. When one considers a different filter function, the dynamics becomes non-markovian and therefore the problem is much more complicated. Basically, this is the issue that for a long time blocked further analytical progress on this problem. In paper I of this series we have developed a formalism in which the problem is formulated in terms of a path integral with boundaries, and non-markovian corrections can be computed perturbatively. In eq. (91) this effect enters through the constant $\tilde{\kappa}$, defined as $\tilde{\kappa} = a\kappa$ with κ given by eq. (4).

The third effect, which was the subject of the present paper, is the inclusion of the non-Gaussianities. These are contained in the function $h_{\rm NG}(\sigma)$. Using the path integral technique developed in paper I, we have computed it to leading, next-to-leading and next-to-next-to-leading order in the parameter σ^2/δ_c^2, which is small for large halo masses, where one can hope to see the effect of non-Gaussianities on the halo mass function. Our result is

$$h_{\rm NG}(\sigma) = \frac{a^2\delta_c^4}{\sigma^4}\mathcal{S}_3(\sigma) - \frac{a\delta_c^2}{\sigma^2}\left(2\mathcal{S}_3(\sigma)+\mathcal{U}_3(\sigma)-\frac{d\mathcal{S}_3}{d\ln\sigma}\right) - \left(\mathcal{S}_3(\sigma)+\mathcal{U}_3(\sigma)+\mathcal{V}_3(\sigma)+\frac{d\mathcal{S}_3}{d\ln\sigma}+\frac{d\mathcal{U}_3}{d\ln\sigma}\right) + \mathcal{O}\left(\frac{\sigma^2}{\delta_c^2}\right). \tag{92}$$

The functions $\mathcal{S}_3(\sigma)$, $\mathcal{U}_3(\sigma)$ and $\mathcal{V}_3(\sigma)$ are defined in terms of the three-point correlator of the smoothed density field $\langle\delta(S_1)\delta(S_2)\delta(S_3)\rangle$ and of its derivatives, as follows,

$$\mathcal{S}_3 = \frac{1}{S^2}\langle\delta^3(S)\rangle\,, \tag{93}$$

$$\mathcal{U}_3 = \frac{3}{S}\left[\frac{d}{dS_1}\langle\delta(S_1)\delta^2(S)\rangle\right]_{S_1=S}, \tag{94}$$

$$\mathcal{V}_3 = \frac{9}{2}\left[\frac{d^2}{dS_1^2}\langle\delta(S_1)\delta^2(S)\rangle\right]_{S_1=S} + 12\left[\frac{d}{dS_1}\frac{d}{dS_2}\langle\delta(S_1)\delta(S_2)\delta(S)\rangle\right]_{S_1=S_2=S}, \tag{95}$$

and we prefer to write them as functions of $\sigma = \sqrt{S}$.

Our result has passed to a good accuracy various comparisons with numerical results. First of all, one can study numerically what happens in the excursion set theory, with fixed (rather than diffusing) barrier and tophat filter in coordinate space, by performing a Monte Carlo realization of the first-crossing distribution of excursion set theory, obtained by integrating numerically a Langevin equation with a colored noise. This was recently performed in detail in Robertson et al. (2008) (see also Bond et al. (1991)). In this limit our analytical result is obtained from eq. (91) setting $a = 1$ (since the barrier is taken as fixed in the Monte Carlo simulation) and $h_{\rm NG}(\sigma) = 0$, i.e. we are testing the effect of κ. Comparing our results in paper I with Fig. 4 of Robertson et al. (2008) we find very good agreement. This is a first useful test of our technique.

Using eq. (91) with $a \simeq 0.80$ (obtained by reading the diffusion coefficient of the barrier D_B from N-body simulations, and using our prediction $a = 1/(1+D_B)$) and with κ given in eq. (4), and setting $h_{\rm NG}(\sigma) = 0$, we can compare our result with the mass function found in N-body simulations with gaussian initial conditions. The comparison is shown in Figs. 6 and 7 of paper II. For all values of $\sigma^{-1} \geq 0.3$ the discrepancy between our analytic result and the Tinker et al. fit to the same N-body simulation is smaller than 20%, and for $\sigma^{-1} \geq 1$ it is smaller than 10%. Considering that our result comes from an analytic model of halo formation with no tunable parameter (the parameter a is fixed once D_B is given, and we do not have the right to tune it), while the Tinker et al. fitting formula is simply a fit to the data with four free parameters, we think that this result is quite encouraging. The numerical accuracy is actually the best that one could have hoped for, considering for instance that we have neglected second-order non-markovian corrections.

Finally, our prediction for the function $h_{\rm NG}(\sigma)$ can be tested against N-body simulations with non-Gaussian initial conditions. To leading order in the small σ limit, our result reduces to that proposed by LoVerde et al. (2008) and Matarrese et al. (2000) using non-Gaussian extensions of PS theory, and it has been found in Grossi et al. (2009) that this formula reproduces very well the data, see in particular their Figs. 6 and 7. The size of the error bars is probably still too large for discriminating between different forms of the subleading term.

We thank Sabino Matarrese for useful discussions. The work of MM is supported by the Fond National Suisse. The work of AR is supported by the European Community's Research Training Networks under contract MRTN-CT-2006-035505.

APPENDIX

A. EXTENSION OF PRESS-SCHECHTER THEORY TO NON-GAUSSIAN FLUCTUATIONS

As we repeatedly emphasized, the really interesting quantity for comparison with experimental data from galaxy surveys, and with N-body simulations, is the distribution function Π_ϵ, that generalizes excursion set theory to non-Gaussian fluctuations. The function $\Pi_{\rm PS}$ defined in eq. (20), where the integrations over the variables $d\delta_i$ run up to $+\infty$ rather than up to δ_c, not only suffers from the fact that it predicts that only a fraction of the total mass of the Universe finally ends up in virialized objects

(the infamous factor of two that Press-Schechter where forced to introduce by hand) but also misses all the subtle correlations between different scales which are just one of the characteristic features of non-Gaussianities. For this reason, in the body of this paper we concentrated on the computation of Π_ϵ. Still, it is interesting to see how our path integral formalism reproduces PS theory and generalizes it to non-Gaussian theories. We discuss the issue in this appendix. In particular, we will see that, even in the non-Gaussian case, $\Pi_{\rm PS}$ satisfies a differential equation which is local in "time", the Kramers-Moyal equation, and which generalizes the Fokker-Planck equation. It is interesting to contrast this result with what happens for Π_ϵ which instead, as discussed in paper I, does not satisfy any local diffusion-like equation.

With our "microscopic" formalism based on the path integral, it is very easy to derive PS theory and to extend it to non-Gaussian fluctuations. Simply, in eq. (29) each integral over $d\delta_i$, with $1\leq i\leq n-1$, produces a factor $2\pi\delta_D(\lambda_i)$, which allows us to perform trivially all the integrals over $d\lambda_i$ with $i<n$. Denoting the residual variable λ_n by λ and setting for notational simplicity $\delta_0=0$ (the general result is recovered with $\delta\to\delta-\delta_0$), eq. (29) becomes

$$\Pi_{\rm PS}(\delta_0=0;\delta;S)=\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\exp\{i\lambda\delta+\sum_{p=2}^{\infty}\frac{(-i\lambda)^p}{p!}\mu_p(S)\}\,. \tag{A1}$$

When all μ_p with $p\geq 3$ vanish, the integral gives a gaussian and we get back the standard PS result,

$$\Pi^{\rm PS}(\delta_0=0;\delta;S)=\frac{1}{(2\pi S)^{1/2}}\,e^{-\delta^2/2S}\,, \tag{A2}$$

since, by definition $\mu_2(S)=S$, where S is the variance computed with the filter function of our choice. Equation (A1) generalizes PS theory to arbitrary non-Gaussian theories.[11] Observe that eqs. (A1) and (A2) hold independently of the filter function used, and the μ_p are the cumulants computed with the filter function in which one is interested.

Equation (A1) is a well-known result in the theory of stochastic processes (see e.g. Risken (1984)), and it was applied to $f_{\rm NL}$-theory in Matarrese et al. (2000). Using this expression, the usual strategy in the literature is to compute $\mathcal{F}(S)$ using

$$\mathcal{F}_{\rm PS}(S)=\frac{\partial}{\partial T}\int_{\delta_c}^{\infty}dx\,\Pi_{\rm PS}(\delta_0;\delta;S)\,, \tag{A3}$$

and to multiply by hand by a fudge factor $\simeq 2$ to ensure the proper normalization. As we have shown in the discussion below eq. (59), this multiplication by a fudge factor is not justified for non-Gaussianities. Still, let us discuss from the mathematical point of view the properties of the function $\Pi_{\rm PS}$, in order to contrast them with the excursion set theory distribution function Π_ϵ.

First of all, it is instructive to rederive the expression (A1) for $\Pi_{\rm PS}$, with generic filter and generic non-Gaussian theory in an alternative way, using the technique developed in paper I for computing the effect of the correction Δ_{ij} to the two-point function, see eqs. (38) and (39). To compute $\Pi_{\rm PS}(\delta_0;\delta;S)$ when the two-point correlator $\langle\delta_i\delta_j\rangle_c$ is generic, rather than equal to $\min(S_i,S_j)$, and in the presence of the higher-order correlators, we write

$$\langle\delta_i\delta_j\rangle_c=\min(S_i,S_j)+[\langle\delta_i\delta_j\rangle_c-\min(S_i,S_j)]\equiv\epsilon A_{ij}+\epsilon B_{ij}\,. \tag{A4}$$

Observe that $\epsilon B_{nn}=\mu_2(S)-S=0$. We then expand the exponential in eq. (29) in powers of ϵB_{ij} and of the higher-order correlators,

$$\begin{aligned}\Pi_{\rm PS}(\delta_0;\delta_n;S_n)&=\int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda\left[1-\frac{1}{2}\sum_{i,j=1}^{n}\lambda_i\lambda_j\epsilon B_{ij}+\frac{(-i)^3}{3!}\sum_{i,j,k=1}^{n}\lambda_i\lambda_j\lambda_k\,\langle\delta_i\delta_j\delta_k\rangle_c+\ldots\right]e^{i\sum_{i=1}^n\lambda_i\delta_i-\frac{1}{2}\sum_{i,j=1}^n\lambda_i\lambda_j\epsilon A_{ij}}\\&=\int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\left[1+\frac{1}{2}\sum_{i,j=1}^{n}\epsilon B_{ij}\partial_i\partial_j-\frac{1}{3!}\sum_{i,j,k=1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\,\partial_i\partial_j\partial_k+\ldots\right]\int\mathcal{D}\lambda\,e^{i\sum_{i=1}^n\lambda_i\delta_i-\frac{1}{2}\sum_{i,j=1}^n\lambda_i\lambda_j\epsilon A_{ij}}\,,\end{aligned} \tag{A5}$$

where $\partial_i=\partial/\partial\delta_i$. The derivatives ∂_i contribute only when the index $i=n$, otherwise we have a total derivative with respect to an integration variable, and the corresponding boundary terms at $\delta=\pm\infty$ term vanish. Here it is crucial that one integrates up to $+\infty$. When we rather consider Π_ϵ, instead of $\Pi_{\rm PS}$, the upper integration limit is δ_c and we remain with complicated and non-local boundary terms, compare e.g. with eq. (83) of paper I. For $\Pi_{\rm PS}$ however this boundary term is absent and

$$\begin{aligned}\Pi_{\rm PS}(\delta_0;\delta_n;S_n)&=\left[1-\frac{1}{3!}\langle\delta_n^3\rangle_c\,\partial_n^3+\ldots\right]\int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\,W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_i,\ldots,\delta_n;S_n)\\&=\left[1-\frac{1}{3!}\langle\delta_n^3\rangle_c\,\partial_n^3+\ldots\right]\Pi^{0,\rm gau}(\delta_0;\delta;S)\,.\end{aligned} \tag{A6}$$

[11] A word of caution is necessary when one considers eq. (A1) with correlators μ_p with $p\geq 4$. For instance, keeping only μ_2, μ_3 and μ_4, one is faced with an integral that diverges, since $\mu_4(S)=\langle\delta^4(S)\rangle>0$. The correct statement is that $\Pi_{\rm PS}(\delta_0;\delta;S)$ is given, order by order, by the expansion of eq. (A1) in powers of μ_4. However, the expansion in powers of μ_4 is only an asymptotic series, which can be used to approximate the true result up to a finite order in μ_4, but diverges if we keep an infinite number of terms. If instead the highest cumulant that we include in eq. (A1) is μ_6, the integral converges because $(-i)^6\mu_6=-\mu_6<0$, while the integral diverges again if the highest cumulant that we include in eq. (A1) is μ_8, since $(-i)^8\mu_8=+\mu_8>0$, and so on. Anyhow, the whole issue of the full resummation of the contributions of the μ_4 or higher-order correlators is physically irrelevant. These correlators are in general computed using phenomenological parametrization of the non-Gaussianities, such as $f_{\rm NL}$-theory, that are meant to be a useful description of the true non-Gaussianities only to leading, and at most next-to-leading order in $f_{\rm NL}$, so in general only the first few terms in the series makes sense physically.

Since the derivative $\partial_n = \partial/\partial\delta_n$ does not act on the correlators $\langle\delta_n^p\rangle$ (which are functions of S_n, but not of δ_n), the expansion in the square brackets can be exponentiated back, and we can write

$$\Pi_{\rm PS}(\delta_0;\delta;S) = e^{\hat{K}_{\rm NG}}\Pi^{0,{\rm gau}}(\delta_0;\delta;S)\,, \tag{A7}$$

where (using now δ_0 generic)

$$\Pi^{0,{\rm gau}}(\delta_0;\delta;S) = \frac{1}{(2\pi S)^{1/2}}\,e^{-(\delta-\delta_0)^2/(2S)}\,, \tag{A8}$$

and the differential operator $\hat{K}_{\rm NG}$ is given by

$$\hat{K}_{\rm NG} = \sum_{p=3}^{\infty}\frac{(-1)^p}{p!}\mu_p(S)\frac{\partial^p}{\partial\delta^p}\,. \tag{A9}$$

To prove the equivalence of eqs. (A7) and (A1) we write eq. (A1) as

$$\Pi_{\rm PS}(\delta_0=0;\delta;S) = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\exp\left\{i\lambda\delta-\frac{1}{2}\mu_2(S)\lambda^2+W_{\rm NG}(\lambda)\right\}\,, \tag{A10}$$

with

$$W_{\rm NG}(\lambda) = \sum_{p=3}^{\infty}\frac{(-i\lambda)^p}{p!}\mu_p(S)\,, \tag{A11}$$

and we expand the exponential in powers of $W_{\rm NG}(\lambda)$. Using $\lambda^p e^{i\lambda x} = (-i\partial_x)^p e^{i\lambda x}$ we see that $W_{\rm NG}(\lambda)e^{i\lambda x} = W(-i\partial_x)e^{i\lambda x}$, and the same holds for any power of $W_{\rm NG}(\lambda)$, so

$$\exp\{W_{\rm NG}(\lambda)\}e^{i\lambda x} = \exp\{W_{\rm NG}(-i\partial_x)\}e^{i\lambda x} = e^{\hat{K}_{\rm NG}}e^{i\lambda x}\,. \tag{A12}$$

Therefore eq. (A10) becomes

$$\Pi_{\rm PS}(\delta_0=0;\delta;S) = e^{\hat{K}_{\rm NG}}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\exp\left\{i\lambda\delta-\frac{1}{2}\mu_2(S)\lambda^2\right\}\,, \tag{A13}$$

which agrees with eq. (A7). So, the distribution function $\Pi_{\rm PS}$ that gives the extension of the PS formalism to non-Gaussian fluctuations can be written equivalently in the integral form (A1) or in the differential form (A7), with $\hat{K}_{\rm NG}$ given by eq. (A9).

It is interesting to observe that the function $\Pi_{\rm PS}(\delta_0=0;\delta;S)$ obeys a local differential equation, both in the gaussian and in the non-Gaussian case. Consider first a gaussian theory with a generic filter function, so $\mu_p(S)=0$ for $p\geq 3$. In order to see exactly where enters the difference between integrating up to δ_c, as in Π_ϵ and integrating up to $+\infty$, we start from the definition

$$\Pi_{\rm PS}(\delta_0;\delta_n;S_n) = \int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda\,e^{i\lambda_i\delta_i-\frac{1}{2}\langle\delta_i\delta_j\rangle\lambda_i\lambda_j}\,, \tag{A14}$$

where the sum over $i,j=1,\ldots,n$ is understood, and we derive a differential equation satisfied by $\Pi_{\rm PS}$, by taking the derivative with respect to S_n,

$$\frac{\partial\Pi_{\rm PS}}{\partial S_n} = \left(-\frac{1}{2}\frac{\partial\langle\delta_k\delta_l\rangle_c}{\partial S_n}\right)\int_{-\infty}^{\infty}d\delta_1\ldots d\delta_{n-1}\int\mathcal{D}\lambda\,\lambda_k\lambda_l\exp\left\{i\lambda_i\delta_i-\frac{1}{2}\langle\delta_i\delta_j\rangle\lambda_i\lambda_j\right\}\,. \tag{A15}$$

Again, using $\lambda\exp\{i\lambda x\} = -i\partial_x\exp\{i\lambda x\}$, inside the integral we can replace $\lambda_k\to -i\partial_k$ and $\lambda_l\to -i\partial_l$. Since we integrate over $d\delta_1,\ldots d\delta_{n-1}$, but not over $d\delta_n$, if $k\neq n$ the term ∂_k, when integrated over $d\delta_k$, is a total derivative and gives zero, because at the boundaries $\delta_k=\pm\infty$ the integrand vanishes exponentially, and the only contribution comes from $k=n$. Similarly, also ∂_l contributes only when $l=n$. This is the step that does not go through for Π_ϵ, when the integration is only up to δ_c, and a complicated boundary term arises, see eq. (83) of paper I. Therefore, since $\langle\delta_n^2\rangle = S_n$, we get a Fokker-Planck equation

$$\frac{\partial\Pi_{\rm PS}}{\partial S} = \frac{1}{2}\frac{\partial^2\Pi_{\rm PS}}{\partial\delta^2}\,, \tag{A16}$$

whose solution, on the line $-\infty<\delta<\infty$, is indeed given by eq. (A2). Equation (A16) can be generalized to the non-Gaussian case using the integral form of the solution (A1) and taking the time derivative,

$$\frac{\partial}{\partial S}\Pi_{\rm PS} = \sum_{p=2}^{\infty}\frac{\dot{\mu}_p(S)}{p!}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}(-i\lambda)^p\exp\left\{i\lambda\delta+\sum_{q=2}^{\infty}\frac{(-i\lambda)^q}{q!}\mu_q(S)\right\}\,, \tag{A17}$$

where $\dot{\mu}_p = d\mu_p/dS$. Inside the integral we can replace $(i\lambda)^p e^{i\lambda x}$ by $\partial_x^p e^{i\lambda x}$, so

$$\frac{\partial\Pi_{\rm PS}}{\partial S} = \sum_{p=2}^{\infty}\frac{(-1)^p}{p!}\dot{\mu}_p(S)\frac{\partial^p\Pi_{\rm PS}}{\partial\delta^p}\,. \tag{A18}$$

This equation is called the Kramers-Moyal (KM) equation or "the stochastic equation", and is well known in the theory of stochastic processes (Stratonovich (1967), Risken (1984)).

In conclusion we have seen that, independently of choice of filter function, $\Pi^{\rm PS}$ satisfies a local differential equation both in the gaussian and in the non-Gaussian case. In the gaussian case it satisfies the FP equation (A16), while in the non-Gaussian case it satisfies the Kramer-Moyal equation (A18). As we already saw in paper I, this is not true for the distribution function Π_ϵ of the excursion set formalism, unless one use a sharp filter in momentum space and the theory is gaussian. Already for gaussian theory and a different filter, we saw in eq. (83) of paper I that the equation satisfied by Π_ϵ, besides the Fokker-Planck operator, contains complicated non-local terms, coming from boundary terms at the upper integration limit δ_c. The same happens, of course, when we include the non-Gaussianities.

B. TERM-BY-TERM COMPUTATION OF $\Pi^{(3,{\rm L})}$

In Section 3.1 we showed that

$$\int_{-\infty}^{\delta_c} d\delta_n\, \Pi^{(3,{\rm L})}_{\epsilon=0}(0;\delta_n;S_n) = \frac{1}{3}\,\frac{\langle\delta_n^3\rangle}{\sqrt{2\pi}\,S_n^{3/2}}\left(1-\frac{\delta_c^2}{S_n}\right)e^{-\delta_c^2/(2S_n)}\,. \tag{B1}$$

Our derivation used the fact that we could replace the sum over i,j,k of $\partial_i\partial_j\partial_k$ in eq. (47) by $\partial^3/\partial\delta_c^3$. It is instructive to reproduce this result by evaluating separately the various terms in the sum, and using the perturbative formalism of paper I. We then split $\sum_{i,j,k=1}^{n}$ into the following terms: (a) $i=j=k=n$. (b) $i<n,\, j=k=n$. (c) $i=j<n, k=n$. (d) $i<j<n, k=n$. (e) $\sum_{i,j,k=1}^{n-1}$, each one with its own combinatorial factor. We denote the corresponding contributions to $\Pi^{(3,{\rm L})}$ as $\Pi^{(3,{\rm La})}$, $\Pi^{(3,{\rm Lb})}$, etc. and, for simplicity, we also use the notation

$$I^{(a)} = \int_{-\infty}^{\delta_c} d\delta_n\, \Pi^{(3,{\rm La})}_{\epsilon=0}(0;\delta_n;S_n)\,, \tag{B2}$$

and so on. As in the computation of the term proportional to $\Delta_{ij}\partial_i\partial_j$ in paper I, we find that the various contributions in this computation can be separately divergent in the continuum limit $\epsilon\to 0$, while their sum is finite, as it clear physically, and as we already know from our derivation in Section 3.1. Indeed, the great virtue of the derivation performed in Section 3.1, using the trick of replacing the sum over $\partial_i\partial_j\partial_k$ with derivatives with respect to δ_c as in eq. (53), is that it directly gives the sum over all combination of indices, thus providing directly the total finite result, and bypassing all problems of divergences that appear if one compute separately the terms corresponding to different combination of indices.

The separate terms can however be computed using the technique developed in paper I, with the finite part prescription. The term $I^{(a)}$ has already been computed in eq. (59), and we have seen that it vanishes. The term (b) is obtained setting $j=k=n$ in eq. (47), and taking into account a combinatorial factor of three corresponding to the three way of choosing which index, among (i,j,k), is not equal to n, so

$$\Pi^{(3,{\rm Lb})}_{\epsilon}(\delta_0;\delta_n;S_n) = -\frac{\langle\delta_n^3\rangle}{2}\sum_{i=1}^{n-1}\partial_n^2\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\partial_i W^{\rm gm} = -\frac{\langle\delta_n^3\rangle}{2}\sum_{i=1}^{n-1}\partial_n^2\left[\Pi^{\rm gm}_\epsilon(\delta_0;\delta_c;S_i)\Pi^{\rm gm}_\epsilon(\delta_c;\delta_n;S_n-S_i)\right]\,. \tag{B3}$$

Using eqs. (65) and (66),

$$I^{(b)} = -\frac{\langle\delta_n^3\rangle}{2\pi}\left[\partial_n\int_0^{S_n} dS_i\,\frac{\delta_c(\delta_c-\delta_n)}{S_i^{3/2}(S_n-S_i)^{3/2}}\exp\left\{-\frac{\delta_c^2}{2S_i}-\frac{(\delta_c-\delta_n)^2}{2(S_n-S_i)}\right\}\right]_{\delta_n=\delta_c} = \frac{\langle\delta_n^3\rangle}{\sqrt{2\pi}\,S_n^{3/2}}\left(1-\frac{\delta_c^2}{S_n}\right)e^{-\delta_c^2/(2S_n)}\,. \tag{B4}$$

The term (c) gives

$$\Pi^{(3,{\rm Lc})}_{\epsilon}(\delta_0;\delta_n;S_n) = -\frac{\langle\delta_n^3\rangle}{2}\sum_{i=1}^{n-1}\partial_n\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\partial_i^2 W^{\rm gm}\,, \tag{B5}$$

so

$$I^{(c)} = -\frac{\langle\delta_n^3\rangle}{2}\sum_{i=1}^{n-1}\left[\partial_i\left(\Pi^{\rm gm}_\epsilon(\delta_0;\delta_c;S_i)\Pi^{\rm gm}_\epsilon(\delta_c;\delta_n;S_n-S_i)\right)\right]_{\delta_n=\delta_c} \tag{B6}$$

This expression is analogous to the one that has already been computed in eqs. (B13)-(B16) of paper I, and it is purely divergent as $1/\sqrt{\epsilon}$, with no finite part, so $\mathcal{FP}[I^{(c)}]=0$.

The term (d) is slightly more complicated, since it requires the α regularization described in appendix B of paper I for extracting the finite part. Setting $i<j<n, k=n$ in eq. (47) and taking into account a combinatorial factor of six, we get

$$\Pi^{(3,{\rm Lc})}_{\epsilon}(\delta_0;\delta_n;S_n) = -\langle\delta_n^3\rangle\partial_n\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\Pi^{\rm gm}_\epsilon(\delta_0;\delta_c;S_i)\Pi^{\rm gm}_\epsilon(\delta_c;\delta_c;S_j-S_i)\Pi^{\rm gm}_\epsilon(\delta_c;\delta_n;S_n-S_j)\,. \tag{B7}$$

We use eqs. (65) and (66), together with $\Pi^{\rm gm}_\epsilon(\delta_c;\delta_c;S)=\epsilon/(\sqrt{2\pi}\,S^{3/2})$, see eq. (112) of paper I, and we get

$$I^{(d)} = -\frac{\langle\delta_n^3\rangle}{\pi\sqrt{2\pi}}\lim_{\delta_n\to\delta_c^-}\int_0^{S_n} dS_i\int_{S_i}^{S_n} dS_j\,\frac{\delta_c(\delta_n-\delta_c)}{S_i^{3/2}(S_j-S_i)^{3/2}(S_n-S_j)^{3/2}}\exp\left\{-\frac{\delta_c^2}{2S_i}-\frac{A^2}{2(S_j-S_i)}-\frac{(\delta_c-\delta_n)^2}{2(S_n-S_j)}\right\}\,, \tag{B8}$$

where $A^2=\alpha\epsilon$ regularizes the integral over dS_j when $S_j\to S_i^+$, and we want to extract the finite part as $A\to 0$. Observe also that here one must be careful not to interchange the limit $\delta_n\to\delta_c^-$ (which comes from the fact that the integral over $d\delta_n$ from $-\infty$ to

δ_c of $\Pi_\epsilon^{(3,\mathrm{Lc})}$ is performed integrating by parts of the derivative ∂_n that appears in eq. (B7)) with the integrals over dS_i and dS_j. The integrals can be carried out using the identity given in eq. (A5) of paper I, and we get

$$I^{(d)} = -\frac{2}{\sqrt{2\pi}}\langle\delta_n^3\rangle\,\frac{1}{S_n^{3/2}}\left(\frac{\delta_c}{A}+1\right)\exp\left\{-\frac{(\delta_c+A)^2}{2S_n}\right\}. \tag{B9}$$

This has a part divergent as $1/A$, i.e. as $1/\sqrt{\epsilon}$, that will combine with the similar divergences from the other terms, a part finite as $A \to 0$, plus terms $\mathcal{O}(A)$ that vanish in the continuum limit. Extracting the finite part we get

$$\mathcal{FP}[I^{(d)}] = -2\,\frac{\langle\delta_n^3\rangle}{\sqrt{2\pi}\,S_n^{3/2}}\left(1-\frac{\delta_c^2}{S_n}\right)e^{-\delta_c^2/(2S_n)}. \tag{B10}$$

Finally, the term (e) can be computed with the by now usual trick of replacing $\sum_{i,j,k=1}^{n-1}$ with $\partial^3/\partial\delta_c^3$, and we get

$$\begin{aligned}\Pi_\epsilon^{(3,\mathrm{Le})}(\delta_0;\delta_n;S_n) &= -\frac{\langle\delta_n^3\rangle}{6}\sum_{i,j,k=1}^{n-1}\int_{-\infty}^{\delta_c} d\delta_1\ldots d\delta_{n-1}\partial_i\partial_j\partial_k W^{\mathrm{gm}} = -\frac{\langle\delta_n^3\rangle}{6}\,\frac{\partial^3}{\partial x_c^3}\Pi_\epsilon^{\mathrm{gm}}(\delta_0;\delta_c;S_n)\\ &= \frac{\langle\delta_n^3\rangle}{6}\,\frac{4\sqrt{2}}{\sqrt{\pi}}\,(2\delta_c-\delta_n)\frac{1}{S_n^{5/2}}\left[3-\frac{(2\delta_c-\delta_n)^2}{S_n}\right]\exp\left\{-\frac{(2\delta_c-\delta_n)^2}{2S_n}\right\},\end{aligned} \tag{B11}$$

and from this we find

$$I^{(e)} = \frac{4}{3}\,\frac{\langle\delta_n^3\rangle}{\sqrt{2\pi}\,S_n^{3/2}}\left(1-\frac{\delta_c^2}{S_n}\right)e^{-\delta_c^2/(2S_n)}. \tag{B12}$$

Summing up eqs. (B4), (B10) and (B12) we get back eq. (B1), as it should.

REFERENCES

Acquaviva, V., Bartolo N., Matarrese S. & Riotto A. 2003, Nucl. Phys. B667, 119.
Afshordi, N. & Tolley, A. 2008, Phys. Rev. D78, 123507.
Bartolo N., Komatsu E., Matarrese S. & Riotto A. 2004, Phys. Rept. 402, 103.
Bartolo N., Matarrese S. & Riotto A. 2005, JCAP 0510, 010.
Bond, J. R., Cole, S., Efstathiou, G. & Kaiser, N. 1991, ApJ. 379, 440.
Carbone C., Verde L., Matarrese S., 2008, ApJ, 684, 1.
Chiu, W. A., Ostriker, J. P., & Strauss, M. A. 1997, ApJ 494, 479.
Dalal N., Dore' O., Huterer D.,Shirokov A., 2008, Phys. Rev. D77, 123514.
Desjacques, V. et al., eprint arXiv:0811.2748.
Grinstein, B., & Wise, M. B. 1986, ApJ, 310, 19.
Grossi, M. et al. 2007, MNRAS 382, 1261.
Grossi, M. et al., arXiv:0902.2013 [astro-ph].
Kang X. et al. 2007, MNRAS 376, , 343.
Komatsu E. et al., 2008, ApJ 180, 330.
Koyama, K., Soda, J., & Taruya, A. 1999, MNRAS, 310, 1111.
LoVerde, M., Miller, A., Shandera, S. & Verde, L. 2008, JCAP 0804, 014.
Lucchin, F., Matarrese, S., & Vittorio, N. 1988, ApJl, 330, L21.
Maggiore, M. & A. Riotto 2009a, arXiv:0903.1249 [astro-ph], (paper I).
Maggiore, M. & A. Riotto 2009b, arXiv:0903.1250 [astro-ph], (paper II).
Maldacena J. 2003, JHEP 0305, 013.
Matarrese, S., Lucchin, F., & Bonometto, S. A. 1986, ApJ., 310, L21.
Matarrese, S., Verde, L. & Jimenez, R. 2000, ApJ 541, 10.
Matarrese, S. & Verde, L. 2008, ApJ., 677, L77.
Moscardini,. L., Matarrese, S., Lucchin, F., & Messina, A. 1991, MNRAS, 248, 424
Peacock, J.A. and Heavens, A.F., 1990, MNRAS 243, 133.
Pillepich, A. Porciani, C. & Hahn, O. 2008, arXiv:0811.4176 [astro-ph].
Press, W. H. & Schechter, P. 1974, ApJ 187, 425.
Risken, H. 1984, "The Fokker-Planck equation", Springer-Verlag, Berlin.
Robertson, B. et al. 2008, arXiv:0812.3148 [astro-ph].
Robinson, J. & Baker, J. E. MNRAS, 311, 781.
Robinson, J., Gawiser, E., & Silk, J. 2000, ApJ, 532, 1.
Sheth, R. K., Mo, H. J., & Tormen, G. 2001, MNRAS, 323, 1.
Sheth, R. K., & Tormen, G. 1999, MNRAS, 308, 119.
Slosar, A., Hirata, C., Seljak, U., Ho, S., Padmanabhan, N. E. 2008, JCAP 08, 031.
Springel, V. *et al.* 2005, Nature 435, 629
Stratonovich, R. L. 1967, "Topics in the Theory of Random Noise", Vol. I and II, Gordon and Breach, New York.
Zentner, A. R. 2007, Int. J. Mod. Phys. D 16 763.